Operational Research and Optimisation

AQ052-3-M-ORO and VD1

Linear Programming

Topic & Structure of the lesson

- Problem Formulation
- Graphical Solution
- Duality, Sensitivity Analysis, Shadow Price
- Excel

Learning Outcomes

- A the end of this topic, You should be able to
  - present a real world problem into a linear programming model.
  - interpret the sensitivity report generated by computer software.

Key Terms you must be able to use

If you have mastered this topic, you should be able to use the following terms correctly in your assignments and exams:

Introduction

- Mathematical programming is used to find the best or optimal solution to a problem that requires a decision or set of decisions about how best to use a set of limited resources to achieve a state goal of objectives.
- **Steps involved in mathematical programming**
  - Conversion of stated problem into a mathematical model that abstracts all the essential elements of the problem.
  - Exploration of different solutions of the problem.
  - Finding out the most suitable or optimum solution.
- **Linear programming** requires that all the mathematical functions in the model be linear functions.

The Linear Programming Model

Let: $X_1, X_2, X_3, \ldots\ldots\ldots, X_n$ = decision variables

Z = Objective function or linear function

Requirement: Maximization of the linear function Z.

$$Z = c_1X_1 + c_2X_2 + c_3X_3 + \ldots\ldots\ldots + c_nX_n \qquad \ldots\ldots \text{Eq (1)}$$

subject to the following constraints:

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &\leq b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &\leq b_2 \\
\vdots \qquad\qquad\qquad & \vdots \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &\leq b_m \\
\text{all } x_j &\geq 0
\end{aligned}$$

ere a_{ij}, b_i, an

The Linear Programming Model

- The linear programming model can be written in more efficient notation as:

Maximize

$$Z = \sum_{j=1}^{n} c_j x_j$$

subject to:

$$\sum_{j=1}^{n} a_{ij} x_j \leq b_i$$

where

$$i = 1, 2, \ldots, m$$

and

$$x_j \geq 0$$

where

$$j = 1, 2, \ldots, n$$

The decision variables, x_l, x_2, ..., x_n, represent levels of *n* competing activities.

Example (1)

A manufacturer would like to maximise the profit by producing standard bags and deluxe bags. A profit of $10 for every standard bag and $9 for every deluxe bag. Every standard bag will require 7/10 hour for cutting and dyeing, ½ hour for sewing, 1 hour finishing time, 1/10 hour for inspection and packaging. Every deluxe bag will require 1 hour for cutting and dyeing, 5/6 hour for sewing, 2/3 hour finishing time, 1/4 hour for inspection and packaging. Director has stated that the time (in hour) given for cutting and dyeing, for sewing, finishing time and for inspection and packaging are 630, 600, 708 and 135 respectively.

Model Formulation

1. What is the objective in words? Maximize the total profit
2. What are the constraints in words?

Cutting & Dyeing Hours Allocated ≤ Cutting & Dyeing Hours Available

Sewing Hours Allocated ≤ Sewing Hours Available

Inspection & Packaging Hours Allocated ≤ Inspection & Packaging Hours Available

Finishing Hours Allocated ≤ Finishing Hours Available

Model Formulation

3. What are the decision variables?

$x1$ = Number of Standards Produced
$x2$ = Number of Deluxes Produced

4. Formulate the objective function:

Profit Contribution Maximize $10\,x1 + 9\,x2$

5. Formulate the constraints:

Subject to:

Cutting & Dyeing:	$7/10\,x1 + x2 \leq 630$
Sewing	$1/2\,x1 + 5/6\,x2 \leq 600$
Finishing	$x1 + 2/3\,x2 \leq 708$
Inspection & Packaging	$1/10\,x1 + 1/4\,x2 \leq 135$

6. Do we need non-negative constraints?

Logic $x1 \geq 0$, $x2 \geq 0$

Example of Solution (Graphical Method)

Enter the linear programming problem here:

- Maximize
- Minimize

$z =$ 10x+9y subject to the constraints:

- Show only the region defined by the following contraints:

```
7/10x+y<=630
1/2x+5/6y<=600
x+2/3y<=708
1/10x+1/4y<=135
```

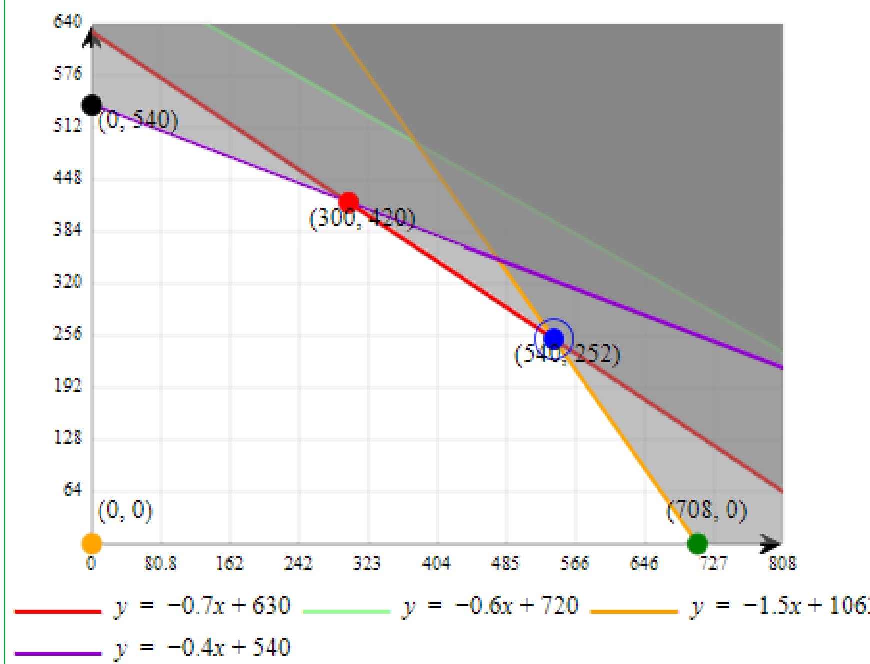
LP Examples | Graphing Examples | Solve

Rounding: 4 decimal places Fraction Mode

Erase Everything

The solution will appear below.

Vertex	Lines through vertex	Value of objective
(540, 252)	$0.7x + y = 630$ $x + 0.6667y = 708$	7668 Maximum
(300, 420)	$0.7x + y = 630$ $0.1x + 0.25y = 135$	6780
(708, 0)	$x + 0.6667y = 708$ $y = 0$	7080
(0, 540)	$0.1x + 0.25y = 135$ $x = 0$	4860
(0, 0)	$x = 0$ $y = 0$	0



Xmin: 0 Xmax: 808

Ymin: 0 Ymax: 640

Gridlines X: 80.8 Y: 650

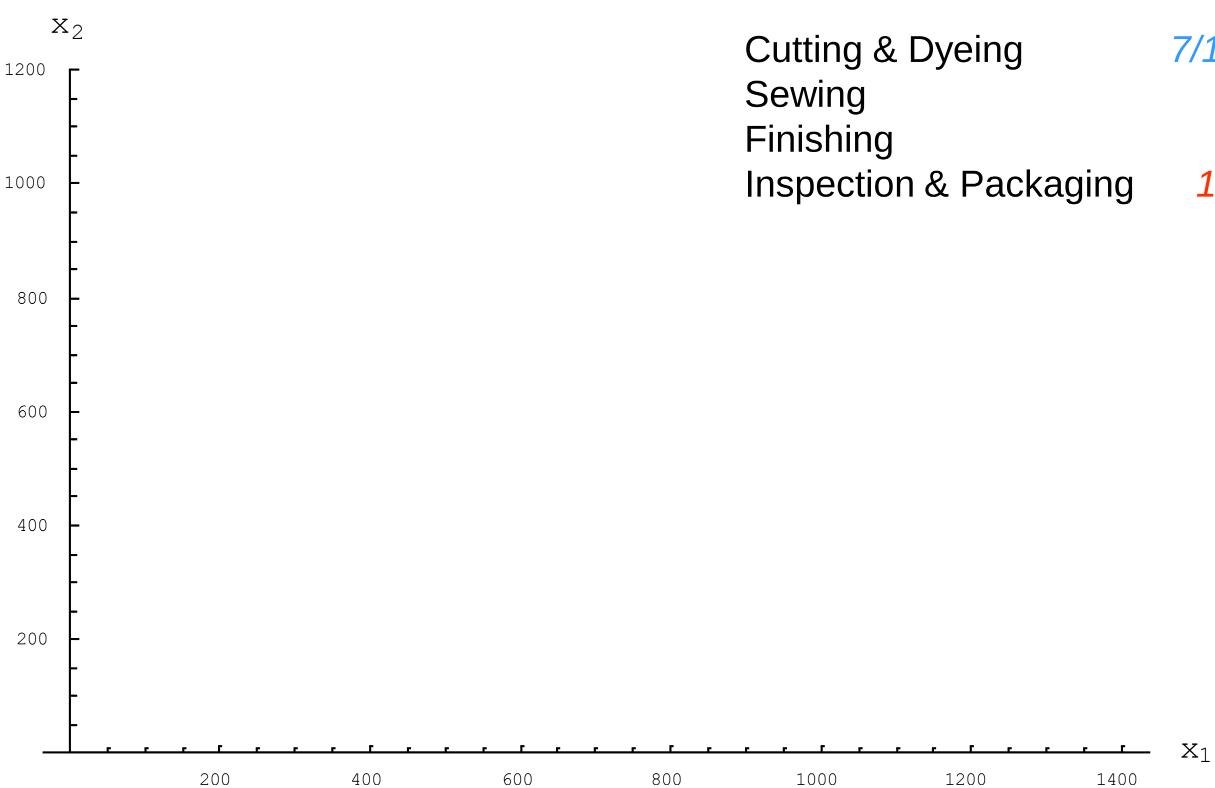
Graph Show vertex coordinates

Source: https://www.zweigmedia.com/utilities/lpg/index.html?lang=en

Constraints Determine Feasible Region

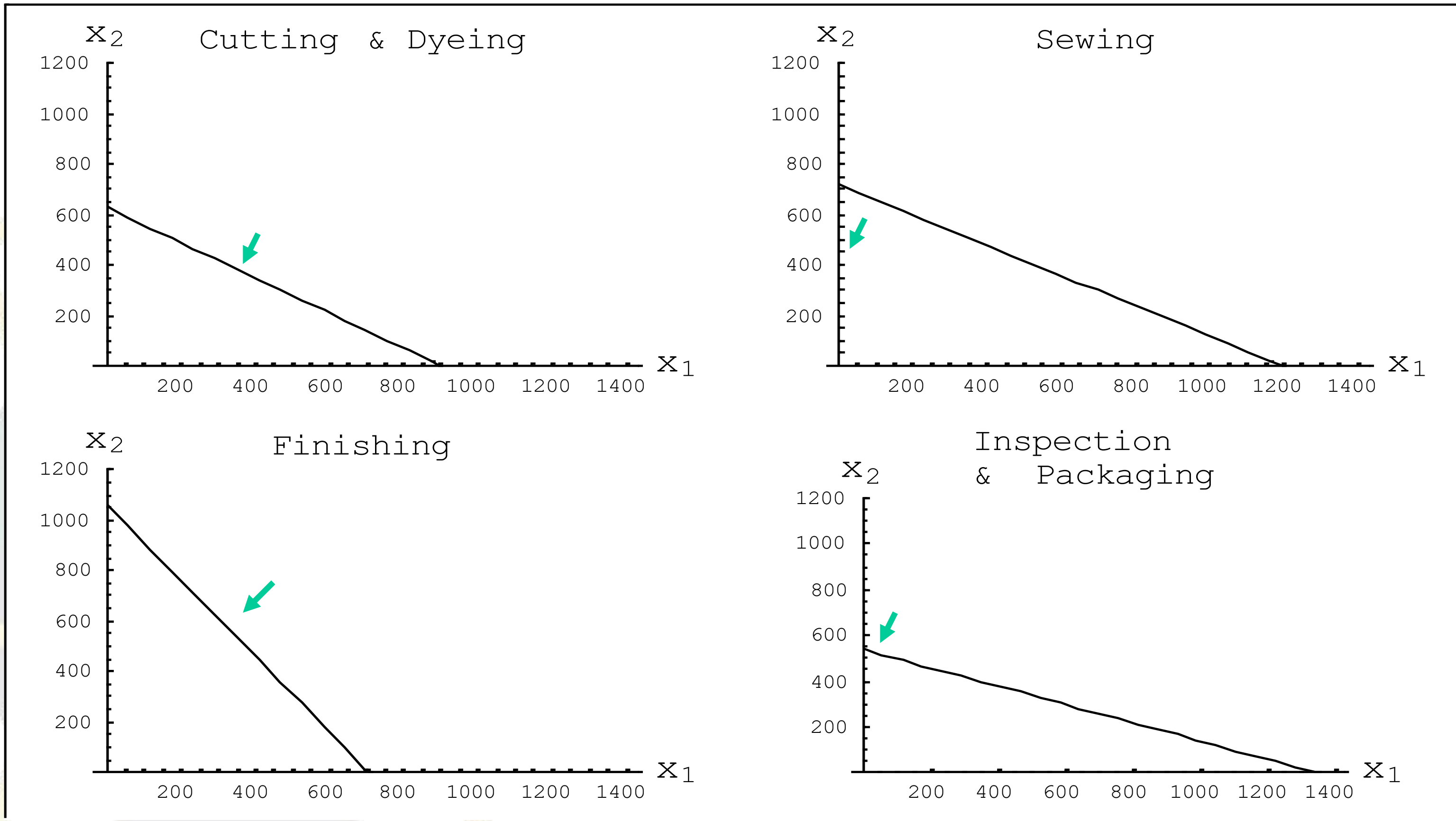
Approach:

1. Draw lines representing the constraints solved as equalities
2. Show the direction of feasibility
3. The feasible region is the set of values which simultaneously satisfies all the constraints

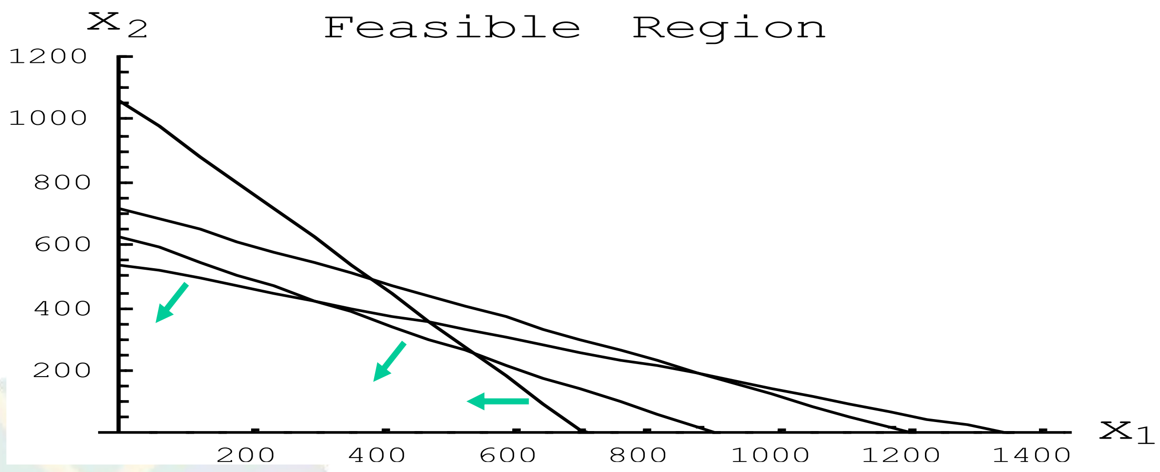


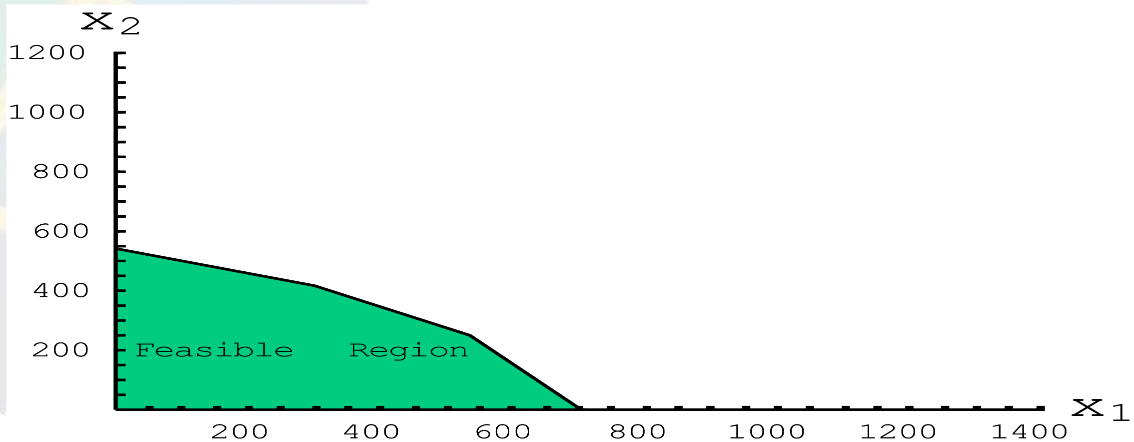
Cutting & Dyeing	$7/10\, x_1 + x_2 \leq 630$
Sewing	$1/2\, x_1 + 5/6\, x_2 \leq 600$
Finishing	$x_1 + 2/3\, x_2 \leq 708$
Inspection & Packaging	$1/10\, x_1 + 1/4\, x_2 \leq 135$
	$x_1 \geq 0, \quad x_2 \geq 0,$

Determine Feasible Region



Determine Feasible Region

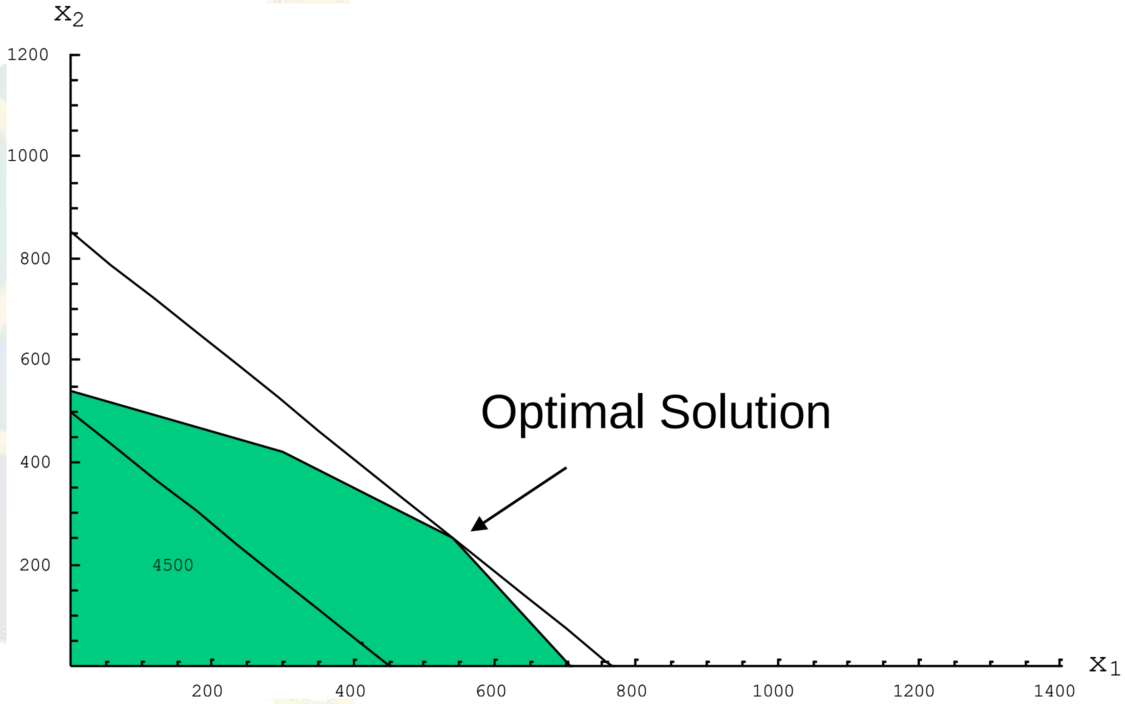




Locate The Optimal Solution - Observations

Observations:

1. The optimal solution cannot fall in the interior of the feasible region
2. The optimal solution must occur at a "corner point" of the feasible region
3. In searching for the optimal solution we only have to examine the corner points



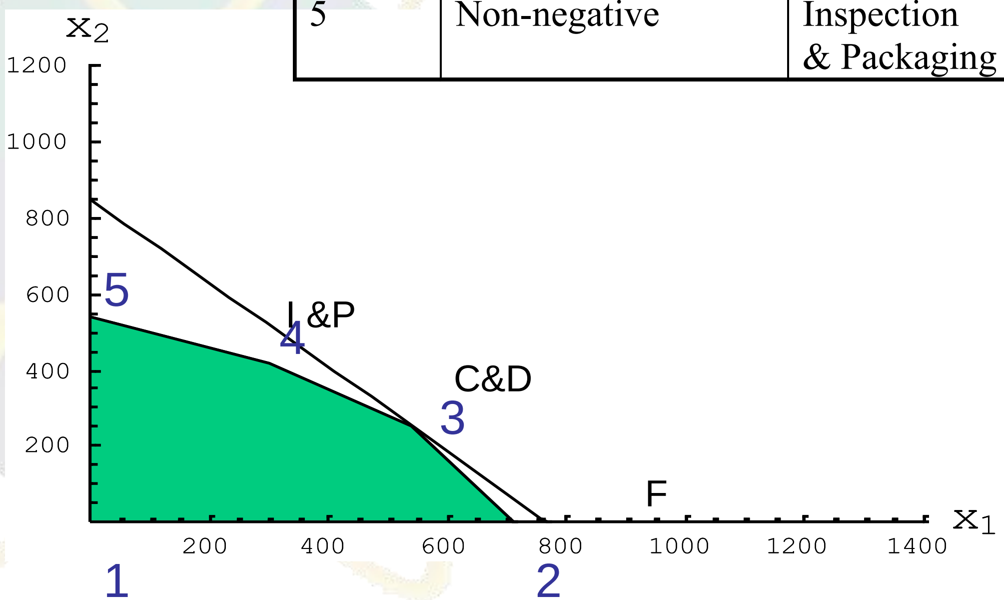
Locate The Optimal Solution

Find extreme points (corner points) and calculate objective values

Extreme Points: the vertices or "corner points" of the feasible region.

With two variables, extreme points are determined by the intersection of the constraint lines.

Point	**Constraint**	**Constraint**	x_1	x_2	**Obj.**
1	Non-negative	Non-negative	0	0	0
2	Non-negative	Finishing	708	0	7080
3	Cutting & Dyeing	Finishing	540	252	7668
4	Cutting & Dyeing	Inspection & Packaging	300	420	6780
5	Non-negative	Inspection & Packaging	0	540	4860



Example 1

Company ABC periodically sponsors public service seminars and programs. Currently, promotional plans are under way for this year's program. Advertising alternatives include television, radio, and newspaper. Audience estimates, costs and maximum media usage limitations are as shown.

	Television	Radio	Newspaper
Audience per advertisement	100,000	18,000	40,000
Cost per advertisement	$2000	$300	$600
Maximum media usage	10	20	10

To ensure a balanced use of advertising media, radio advertisements must not exceed 50% of the total number of advertisements. In addition, television should account for at least 10% of the total number of advertisements. If the promotional budget is limited to $18,200 and the company would like to maximize the total audience contact, what is the allocation of the budget among the three media, and what is the total audience reached?

Exercise: Linear Programming Model

CPI company manufactures a standard dining chair used in restaurant. The demand forecasts for chairs for Q1 and Q2 are 3700 and 4200, respectively.

The chair contains an upholstered seat that can be produced by CPI or purchased from D1.

D1 currently charges $12.5 per seat, but has announced a new price of $13.75 effective the second quarter. CPI can produce at most 3800 seats per quarter at a cost of $10.25 per seat.

Seats produced or purchased in Q1 can be stored in order to satisfy demand in Q2. A seat cost CPI $1.50 each to hold an inventory, and maximum inventory cannot exceed 300 seats. Find the optimal make-or-buy plan for CPI company.

Implementing an LP Model in a Spreadsheet

- Organize the data for the model on the spreadsheet including:
  - Objective Coefficients
  - Constraint Coefficients
  - Right-Hand-Side (RHS) values
    - clearly label data and information
    - visualize a logical layout
    - row and column structures of the data are a good start

Implementing an LP Model in a Spreadsheet

- Reserve separate cells to represent each decision variable
  - arrange them such that the structure parallels the input data
  - keep them together in the same place
  - Right-Hand-Side (RHS) values
- Create a formula in a cell that corresponds to the Objective Function
- For each constraint, create a formula that corresponds to the left-hand-side (LHS) of the constraint
- NOTE: Once you've formulated a spreadsheet model for a certain type of application, it's useful to save it as a template for future use!

Solver Terminology

- Target cell
  - Represents the objective function
  - Must indicate max or min
- Changing cells
  - Represents the decision variables
- Constraint cells
  - the cells in the spreadsheet that represent the *LHS formulas* of the constraints in the model
  - the cells in the spreadsheet that represent the *RHS values* of the constraints in the model
  - TIP: If you scale the constraints such that they are all "≤" or all "≥", then it's easier to input the constraints as a block. Then, you avoid having to enter each constraint as a separate line.

Solver Example: Par, Inc. Model

	A	B	C	D	E	F
1		**Production Time (hrs/unit)**				
2	**Operation**	Standard Bag	Deluxe Bag	Hrs Used		Hrs Available
3	**Cutting & Dyeing**	0.7	1	0	<=	630
4	**Sewing**	0.5	0.83333333	0	<=	600
5	**Finishing**	1	0.66666667	0	<=	708
6	**Inspection & Packaging**	0.1	0.25	0	<=	135
7						
8						**Total Profit**
9	**Profit per Unit**	10	9			0
10						
11	**No. of Units Produced**	0	0			

	A	B	C	D	E	F
1		**Production Time (hrs/unit)**				
2	**Operation**	Standard Bag	Deluxe Bag	Hrs Used		Hrs Available
3	**Cutting & Dyeing**	0.7	1	=B3*$B$11+C3*$C$11	<=	630
4	**Sewing**	0.5	=5/6	=B4*$B$11+C4*$C$11	<=	600
5	**Finishing**	1	=2/3	=B5*$B$11+C5*$C$11	<=	708
6	**Inspection & Packaging**	0.1	0.25	=B6*$B$11+C6*$C$11	<=	135
7						
8						**Total Profit**
9	**Profit per Unit**	10	9			=B9*B11+C9*C11
10						
11	**No. of Units Produced**	0	0			

Answer Report

Solver Parameters: Objective: MAX F9
Variables: B11:C11
Constraints: D3:D6 <= F3:F6
Options: Assume Linear Model
Assume Non-Negative

Worksheet: [LP.xls]Par
Report Created: 1/27/00 11:09:15 PM

Target Cell (Max)

Cell	Name	Original Value	Final Value
$F$9	Profit per Unit Total Profit	0	7668

Optimal objective value

Adjustable Cells

Cell	Name	Original Value	Final Value
$B$11	No. of Untis Produced Standard Bag	0	540
$C$11	No. of Untis Produced Deluxe Bag	0	252

Optimal solution

Constraints

Cell	Name	Cell Value	Formula	Status	Slack
$D$3	Cutting & Dyeing Hrs Used	630	$D$3<=$F$3	Binding	0
$D$4	Sewing Hrs Used	480	$D$4<=$F$4	Not Binding	120
$D$5	Finishing Hrs Used	708	$D$5<=$F$5	Binding	0
$D$6	Inspection & Packaging Hrs Used	117	$D$6<=$F$6	Not Binding	18

Constraints

Binding & Slack Information

Summary of Excel Solver Procedure

					LHS	Inequality	RHS
Constraints	**Left Hand Side Coefficients**				**Formulas**	= >= <=	**Right Hand Side Coefficients**
Coefficients of Decision Variables							**Objective Function**
Decision Variables							

The Role of Sensitivity Analysis of the Optimal Solution

- Is the optimal solution sensitive to changes in input parameters?
- Possible reasons for asking this question:
  - Parameter values used were only best estimates.
  - Dynamic environment may cause changes.
  - “What-if” analysis may provide economical and operational information.

Sensitivity Analysis of Objective Function Coefficients.

- Range of Optimality
  - The optimal solution will remain unchanged as long as
    - An objective function coefficient lies within its ***range of optimality***
    - There are no changes in any other input parameters.
  - The value of the objective function will change if the coefficient multiplies a variable whose value is nonzero.

- Reduced cost

  Assuming there are no other changes to the input parameters, the reduced cost for a variable X_j that has a value of "0" at the optimal solution is:

  – The negative of the objective coefficient increase of the variable X_j ($-\Delta C_j$) necessary for the variable to be positive in the optimal solution

  – Alternatively, it is the change in the objective value per unit increase of X_j.

- Complementary slackness

  At the optimal solution, either the value of a variable is zero, or its reduced cost is 0.

Sensitivity Analysis of Right-Hand Side Values

- In sensitivity analysis of right-hand sides of constraints we are interested in the following questions:
  - Keeping all other factors the same, how much would the optimal value of the objective function (for example, the profit) change if the right-hand side of a constraint changed by one unit?
  - For how many additional or fewer units will this per unit change be valid?

Sensitivity Analysis of Right-Hand Side Values

- Any change to the right hand side of a binding constraint will change the optimal solution.

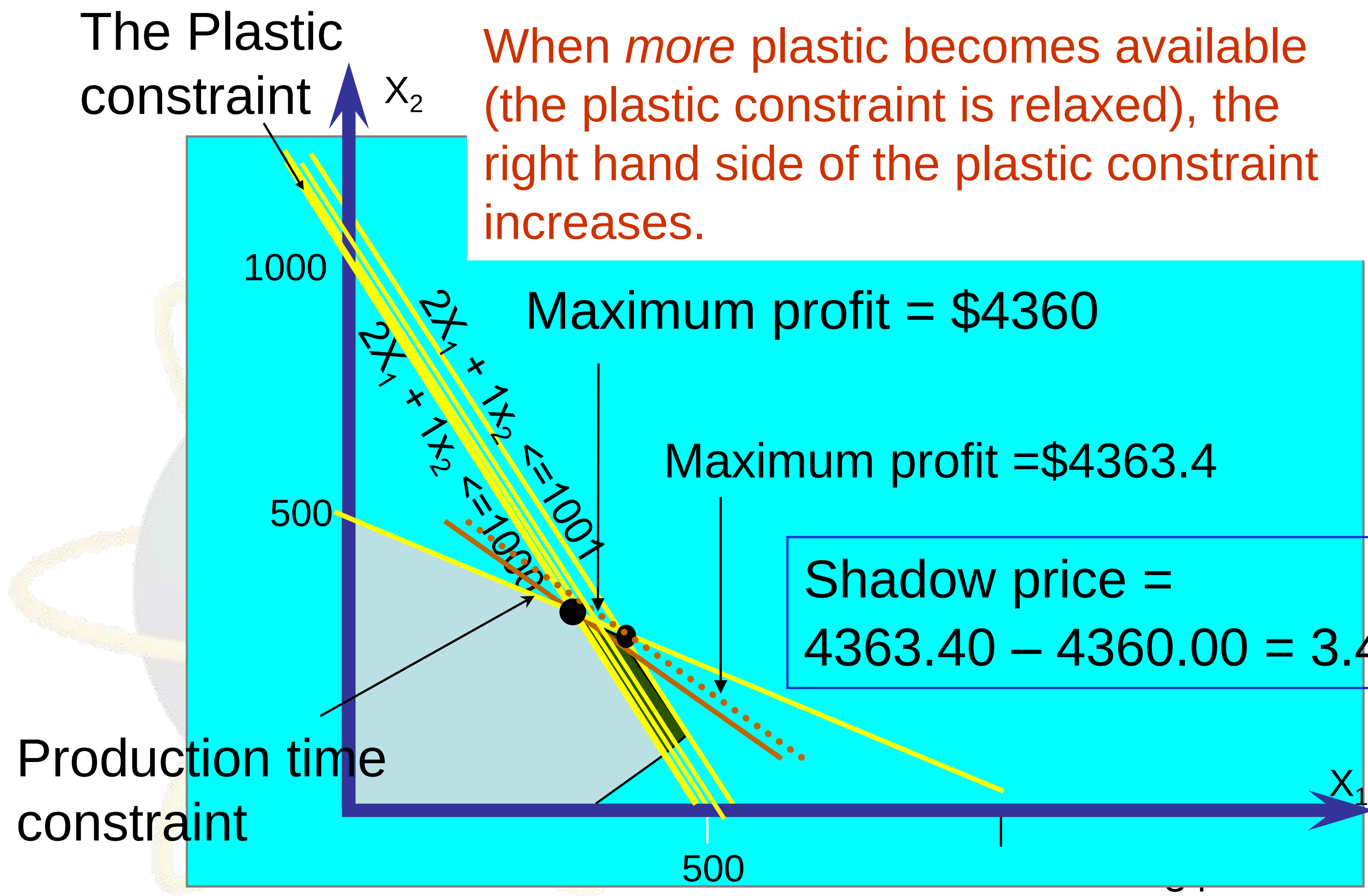
- Any change to the right-hand side of a non-binding constraint that is less than its slack or surplus, will cause no change in the optimal solution.

Shadow Prices

- Assuming there are no other changes to the input parameters, the change to the objective function value per unit increase to a right hand side of a constraint is called the “Shadow Price”

Shadow Price – graphical demonstration

When *more* plastic becomes available (the plastic constraint is relaxed), the right hand side of the plastic constraint increases.



Shadow price = 4363.40 – 4360.00 = 3.40

Range of Feasibility

- Assuming there are no other changes to the input parameters, the range of feasibility is
  - The range of values for a right hand side of a constraint, in which the shadow prices for the constraints remain unchanged.
  - In the range of feasibility the objective function value changes as follows:
    Change in objective value =
    [Shadow price][Change in the right hand side value]

Solver Example: Par, Inc. Model

	A	B	C	D	E	F
1		Production Time (hrs/unit)				
2	Operation	Standard Bag	Deluxe Bag	Hrs Used		Hrs Available
3	Cutting & Dyeing	0.7	1	0	<=	630
4	Sewing	0.5	0.83333333	0	<=	600
5	Finishing	1	0.66666667	0	<=	708
6	Inspection & Packaging	0.1	0.25	0	<=	135
7						
8						Total Profit
9	Profit per Unit	10	9			0
10						
11	No. of Units Produced	0	0			

	A	B	C	D	E	F
1		Production Time (hrs/unit)				
2	Operation	Standard Bag	Deluxe Bag	Hrs Used		Hrs Available
3	Cutting & Dyeing	0.7	1	=B3*$B$11+C3*$C$11	<=	630
4	Sewing	0.5	=5/6	=B4*$B$11+C4*$C$11	<=	600
5	Finishing	1	=2/3	=B5*$B$11+C5*$C$11	<=	708
6	Inspection & Packaging	0.1	0.25	=B6*$B$11+C6*$C$11	<=	135
7						
8						Total Profit
9	Profit per Unit	10	9			=B9*B11+C9*C11
10						
11	No. of Units Produced	0	0			

Interpretation of Solver Output

Constraints

Cell	Name	Cell Value	Formula	Status	Slack
$D$3	Cutting	630	$D$3<=$E$3	Binding	0
$D$4	Sewing	480	$D$4<=$E$4	Not Binding	120
$D$5	Finishing	708	$D$5<=$E$5	Binding	0
$D$6	Inspection	117	$D$6<=$E$6	Not Binding	18

Slack value

Variable Cells

Cell	Name	Final Value	Reduced Cost	Objective Coefficient	Allowable Increase	Allowable Decrease
$B$9	No. of Units Standard Bag	540	0	10	3.5	3.7
$C$9	No. of Units Deluxe Bag	252	0	9	5.285714286	2.333333333

Range of Optimality

Shadow Price

Constraints

Cell	Name	Final Value	Shadow Price	Constraint R.H. Side	Allowable Increase	Allowable Decrease
$D$3	Cutting	630	4.375	630	52.36363636	134.4
$D$4	Sewing	480	0	600	1E+30	120
$D$5	Finishing	708	6.9375	708	192	128
$D$6	Inspection	117	0	135	1E+30	18

Range of Feasibility

Exercise

M&D chemicals produces two products that are sold as raw materials. Management has specified that the combination production for product 1 and 2 must total at least 350 gallons. Separately, a major customer's order for 125 gallons of product 1 must also be satisfied. Product 1 requires 2 hours of processing time per gallon while product 2 requires 1 hour of processing time per gallon, and for the coming month, 600 hours of processing time are available. M&D's objective is to satisfy the above requirements at a minimum total production cost. Production costs are \$2 per gallon for product 1 and \$3 per gallon for product 2.

Generate the sensitivity report and interpret the output.

Quick Review Question

Follow Up Assignment

Summary of Main Teaching Points

Question and Answer Session

Q & A

Next Lesson

Transportation Models