Operational Research and Optimisation

AQ052-3-M-ORO and VD1

Nonlinear Programming

Topic & Structure of the lesson

- Unconstrained algorithms
- Constrained algorithms

Learning Outcomes

- **A the end of this topic, You should be able to apply non-linear programming model in a real world problem.**

Key Terms you must be able to use

If you have mastered this topic, you should be able to use the following terms correctly in your assignments and exams:

Overview

- Many business problems can be modeled only with nonlinear functions.
- Problems that fit the general linear programming format but contain nonlinear functions are termed nonlinear programming (NLP) problems.
- Solution methods are more complex than linear programming methods.
- Often difficult, if not impossible, to determine optimal solution.
- Solution techniques generally involve searching a solution surface for high or low points requiring the use of advanced mathematics.
- Computer techniques (Excel) are used in this chapter.

Optimal Value of a Single Nonlinear Function Basic Model

Profit function, Z, with volume independent of price:

$$Z = vp - c_f - vc_v$$

where v = sales volume

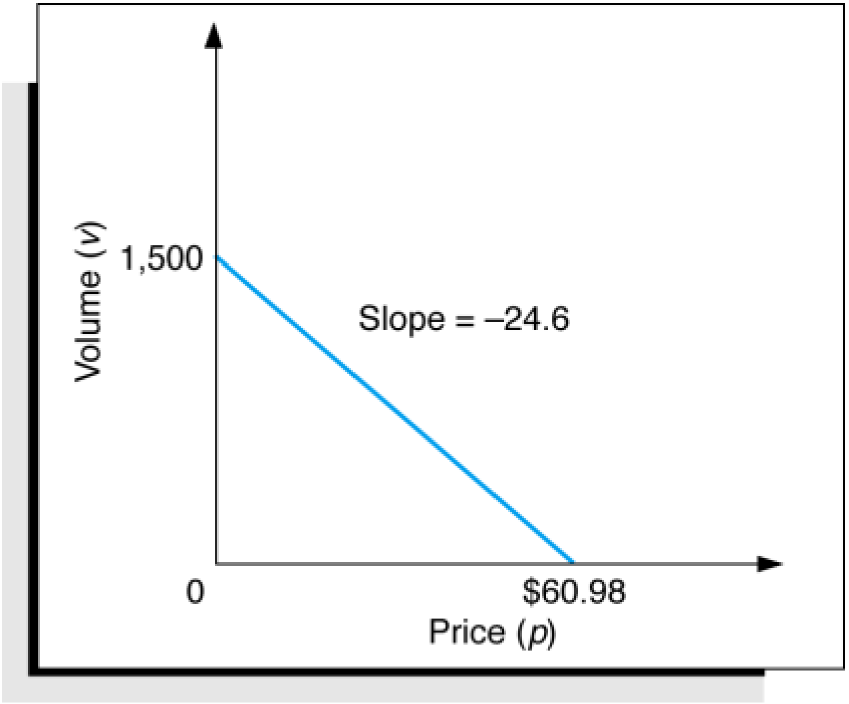
p = price

c_f = fixed cost

c_v = variable cost

Add volume/price relationship:

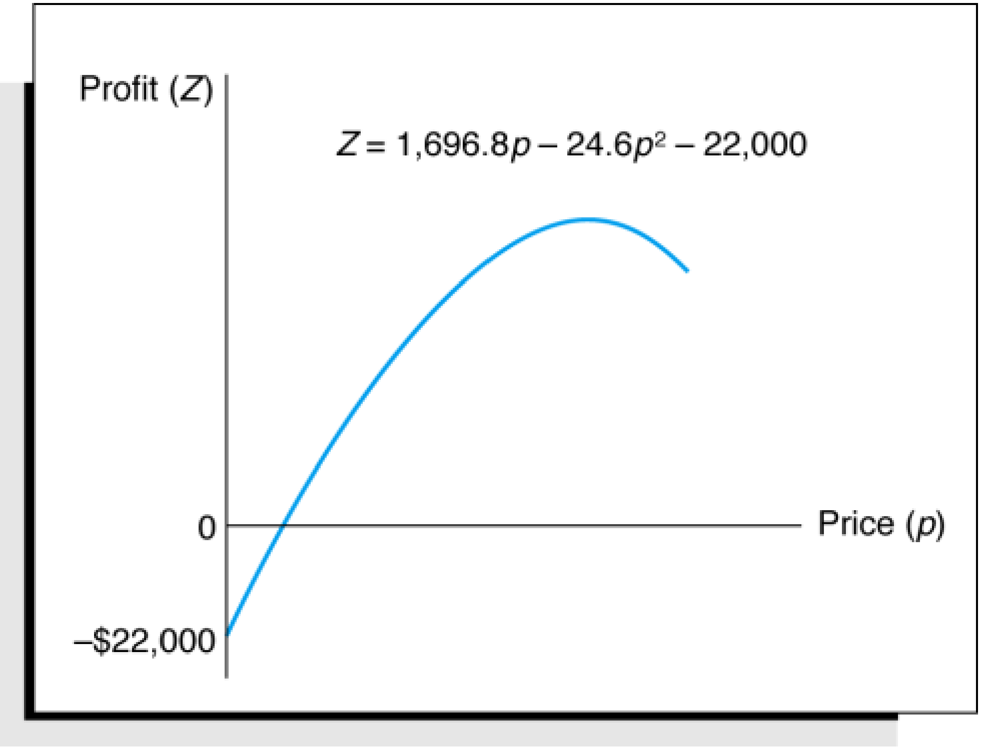
$$v = 1{,}500 - 24.6p$$



Optimal Value of a Single Nonlinear Function Expanding the Basic Model to a Nonlinear Model

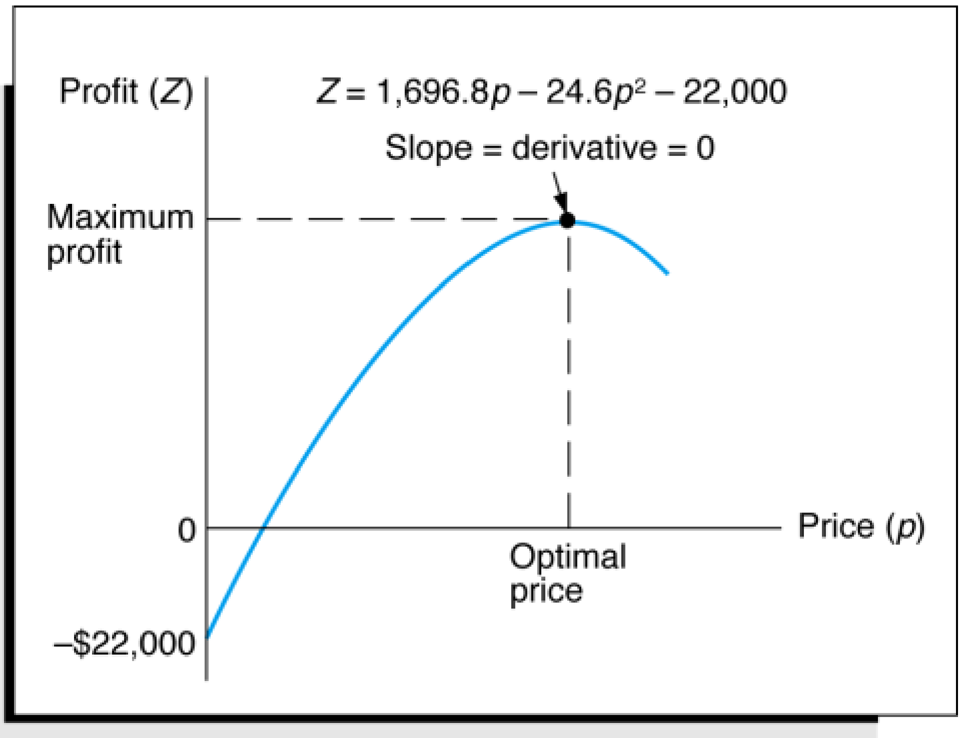
With fixed cost (c_f = \$10,000) and variable cost (c_v = \$8):

$Z = 1{,}696.8p - 24.6p^2 - 22{,}000$



Optimal Value of a Single Nonlinear Function
Maximum Point on a Curve

- The slope of a curve at any point is equal to the derivative of the curve's function.
- The slope of a curve at its highest point equals zero.



Optimal Value of a Single Nonlinear Function Solution Using Calculus

nonlinearity!

$Z = 1{,}696.8 \cdot p - 24.6 \cdot p^2 - 22{,}000$

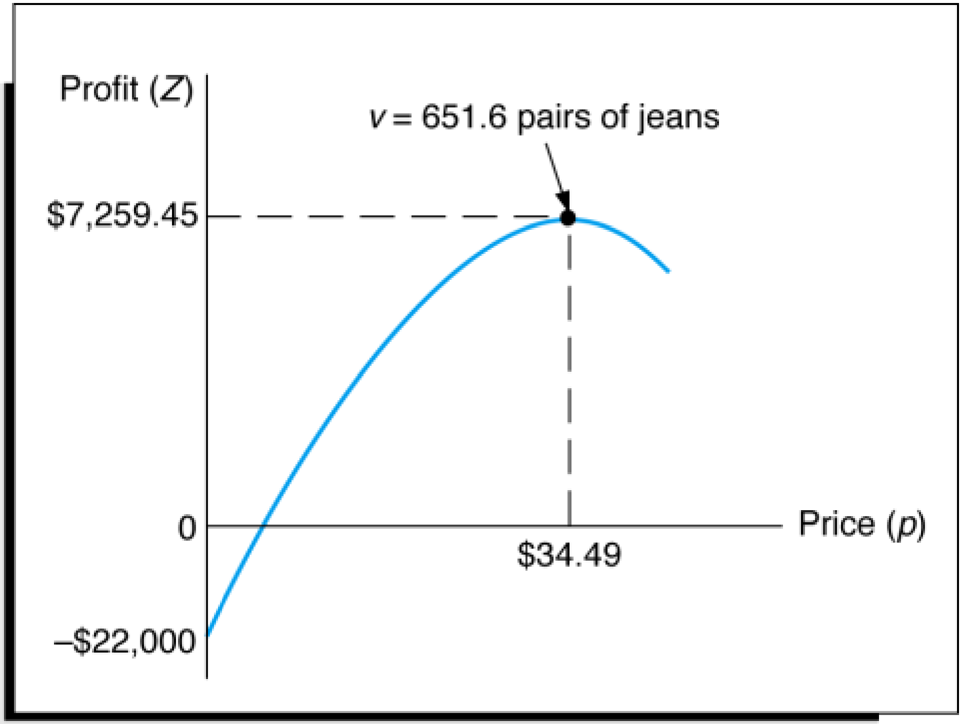
$dZ/dp = 1{,}696.8 - 49.2p$

$p = \$34.49$

$v = 1{,}500 - 24.6p$

$v = 651.6$ pairs of jeans

$Z = \$7{,}259.45$

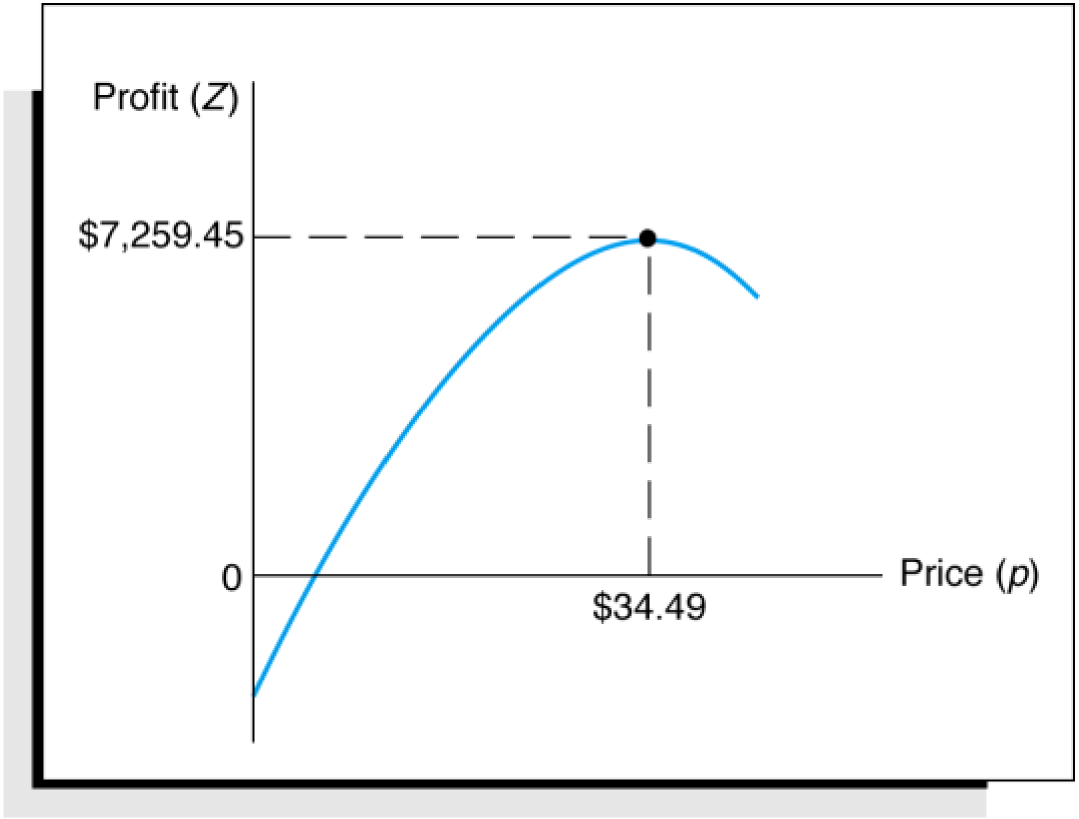


Constrained Optimization in Nonlinear Problems
Definition

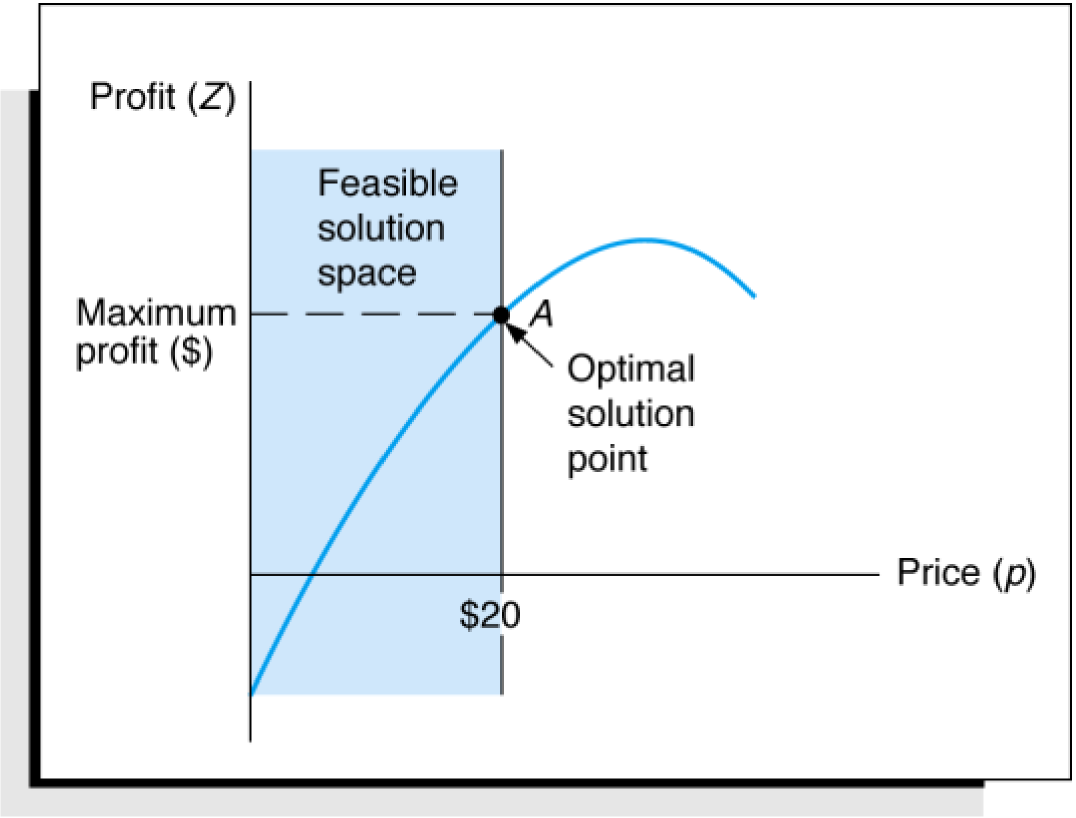
- If a nonlinear problem contains one or more constraints it becomes a constrained optimization model or a *nonlinear programming* model.
- A nonlinear programming model has the same general form as the linear programming model except that the objective function *and/or* the constraint(s) are nonlinear.
- Solution procedures are much more complex and no guaranteed procedure exists.

Constrained Optimization in Nonlinear Problems
Graphical Interpretation (1 of 3)

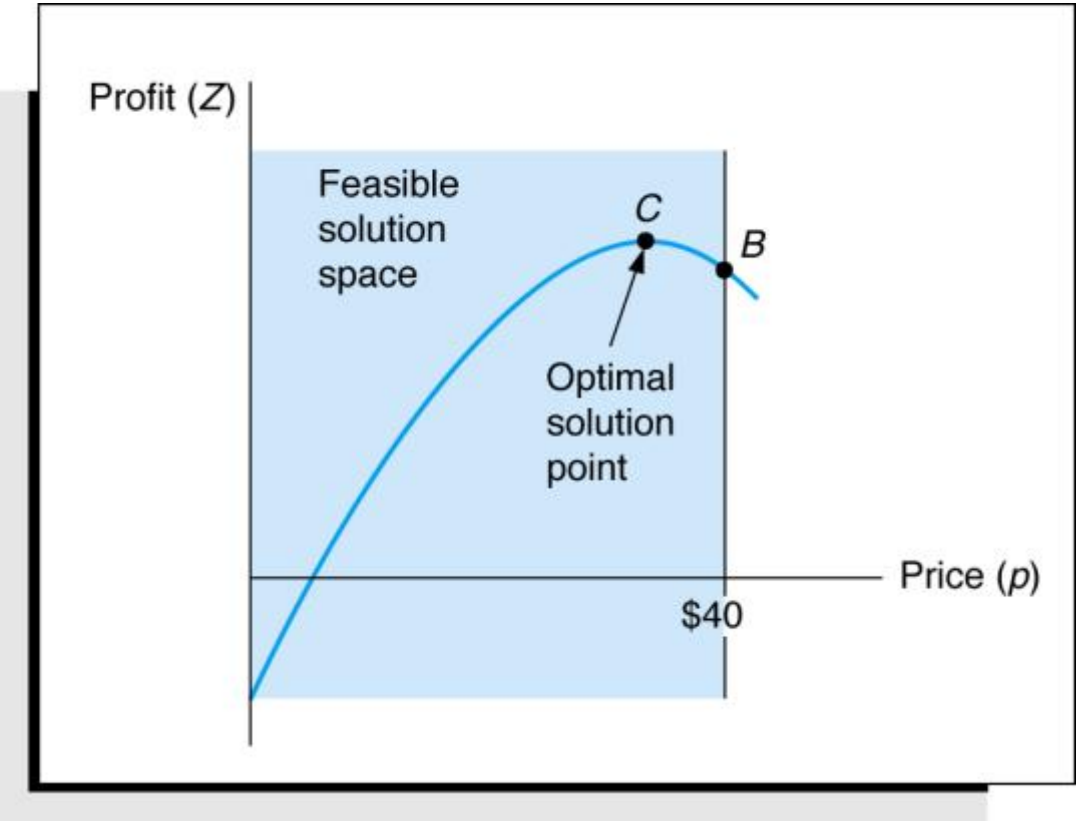
- Effect of adding constraints to nonlinear problem:



Constrained Optimization in Nonlinear Problems
Graphical Interpretation (2 of 3)



Constrained Optimization in Nonlinear Problems
Graphical Interpretation (3 of 3)



Constrained Optimization in Nonlinear Problems
Characteristics

- Unlike linear programming, solution is often not on the boundary of the feasible solution space.
- Cannot simply look at points on the solution space boundary but must consider other points on the surface of the objective function.
- This greatly complicates solution approaches.
- Solution techniques can be very complex.

Western Clothing Company Problem Solution Using Excel

Maximize $Z = (p_1 - 12)x_1 + (p_2 - 9)x_2$

subject to:

$$2x_1 + 2.7x_2 \leq 6,00$$
$$3.6x_1 + 2.9x_2 \leq 8,500$$
$$7.2x_1 + 8.5x_2 \leq 15,000$$

where:

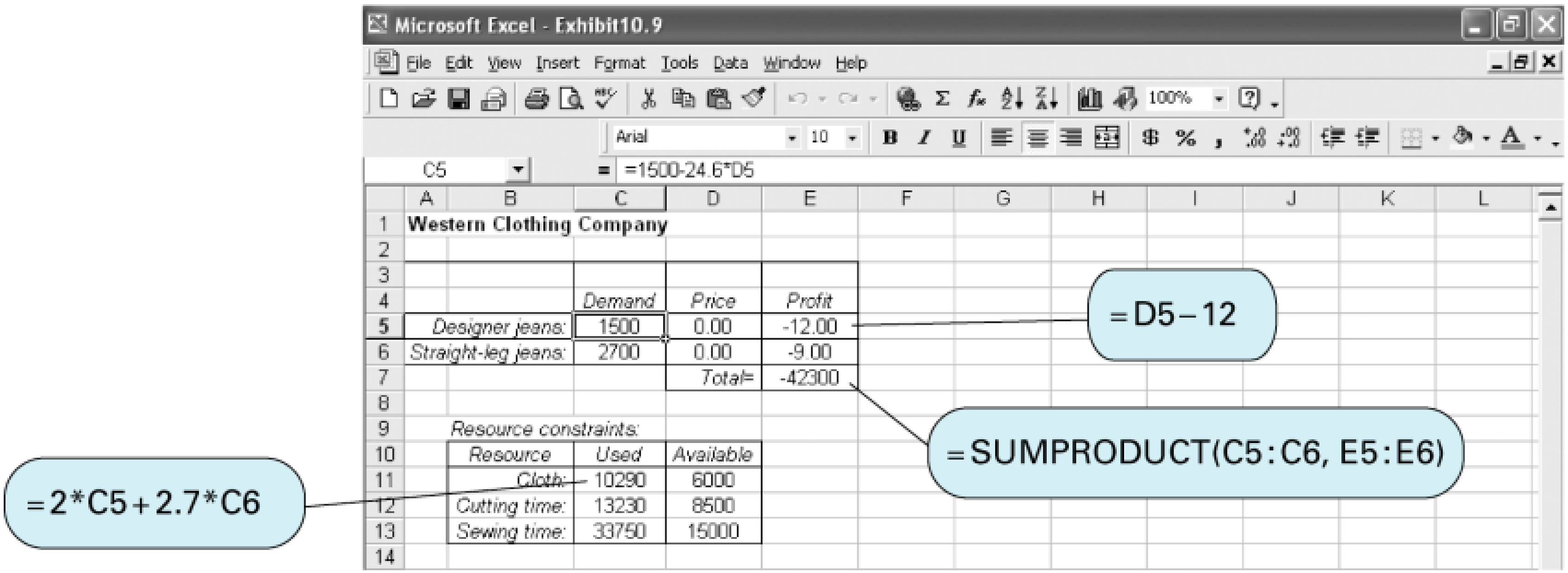
$x_1 = 1,500 - 24.6p_1$

$x_2 = 2,700 - 63.8p$

p_1 = price of designer jeans

p_2 = price of straight jeans

Western Clothing Company Problem Solution Using Excel



Microsoft Excel - Exhibit10.9

C5 = =1500-24.6*D5

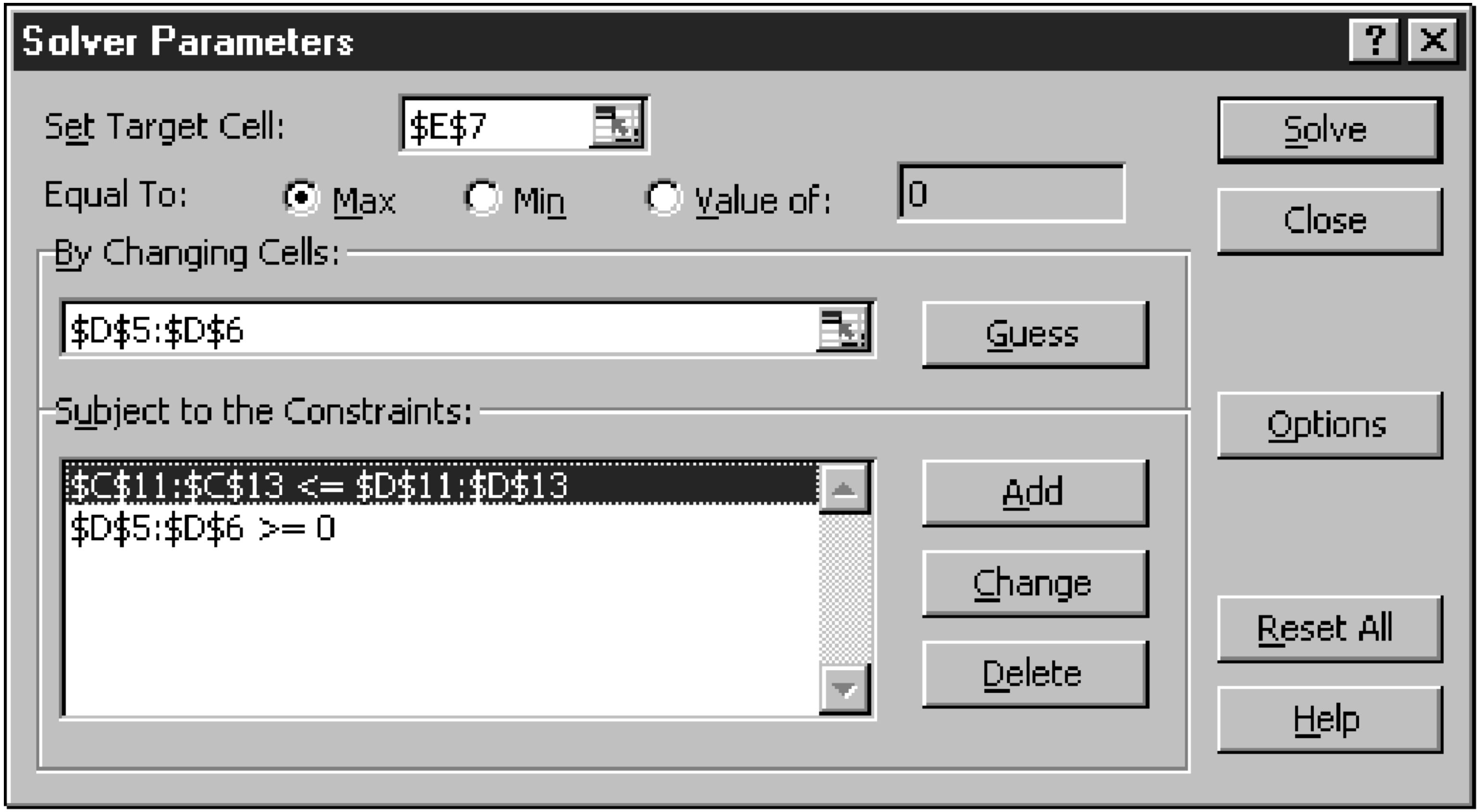
Western Clothing Company

	Demand	Price	Profit
Designer jeans:	1500	0.00	-12.00
Straight-leg jeans:	2700	0.00	-9.00
		Total=	-42300

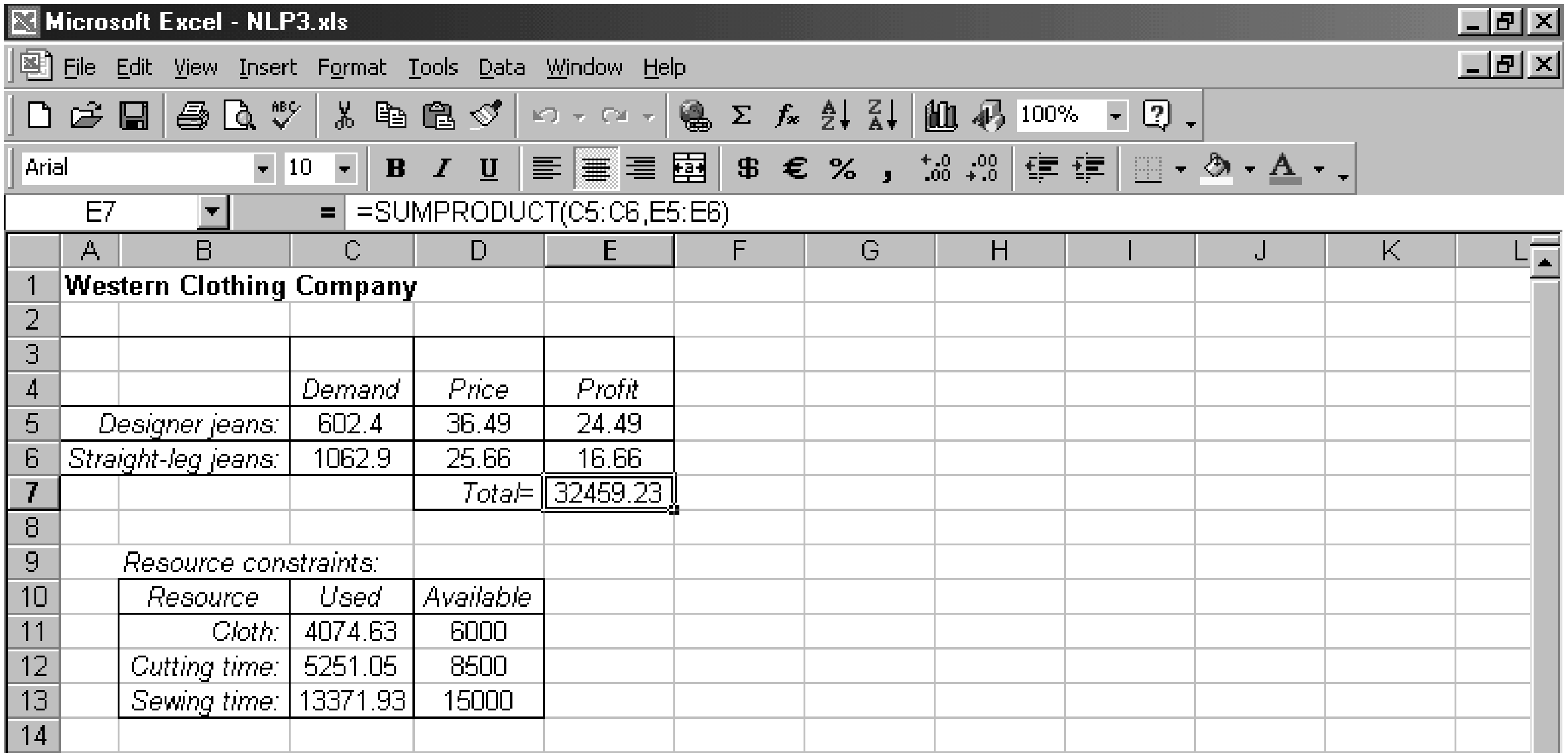
Resource constraints:

Resource	*Used*	*Available*
Cloth:	10290	6000
Cutting time:	13230	8500
Sewing time:	33750	15000

Western Clothing Company Problem Solution Using Excel



Western Clothing Company Problem Solution Using Excel



Microsoft Excel - NLP3.xls

E7 = =SUMPRODUCT(C5:C6,E5:E6)

Western Clothing Company

	Demand	Price	Profit
Designer jeans:	602.4	36.49	24.49
Straight-leg jeans:	1062.9	25.66	16.66
		Total=	32459.23

Resource constraints:

Resource	*Used*	*Available*
Cloth:	4074.63	6000
Cutting time:	5251.05	8500
Sewing time:	13371.93	15000

Facility Location Example Problem Problem Definition and Data (1 of 2)

Centrally locate a facility that serves several customers or other facilities in order to minimize distance or miles traveled (d) between facility and customers.

$$d_i = \text{sqrt}((x_i - x)^2 + (y_i - y)^2) \quad (= \text{straight-line distance})$$

Where:

(x,y) = coordinates of proposed facility

(x_i,y_i) = coordinates of customer or location facility i

Minimize total miles $d = \Sigma\, d_i t_i$

Where:

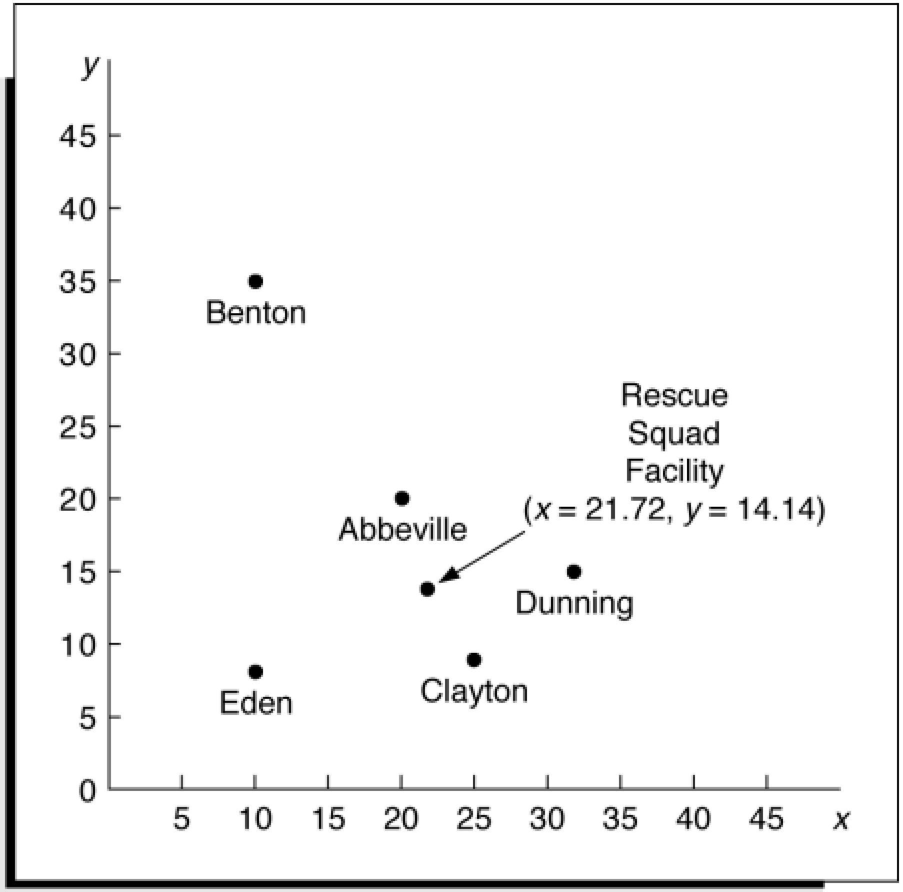
d_i = distance to town i

t_i =annual trips to town i

Facility Location Example Problem
Problem Definition and Data (2 of 2)

	Coordinates		
Town	x	y	Annual Trips
Abbeville	20	20	75
Benton	10	35	105
Clayton	25	9	135
Dunnig	32	15	60
Eden	10	8	90

Facility Location Example Problem Solution Map



Investment Portfolio Selection Example Problem
Definition and Model Formulation (1 of 2)

- Objective of the portfolio selection model is to minimize some measure of portfolio risk (variance in the return on investment) while achieving some specified minimum return on the total portfolio investment.
- Since variance is the sum of squares of differences, it is *mathematically identical* to the “straight-line distance”! Thus, it is possible to visualize variances as such distances, and minimizing the overall variance is then *mathematically identical* to minimizing such distances.

Investment Portfolio Selection Example Problem Definition and Model Formulation (2 of 2)

Minimize $S = \underbrace{x_1^2 s_1^2 + x_2^2 s_2^2 + \ldots + x_n^2 s_n^2}_{\text{“straight-line distance”}} + \Sigma x_i x_j r_{ij} s_i s_j$

where:

S = variance of annual return of the portfolio

x_i, x_j = the proportion of money invested in investments i or j

s_i^2 = the variance for investment i

r_{ij} = the correlation between returns on investments i and j

s_i, s_j = the std. dev. of returns for investments i and j

subject to:

$r_1x_1 + r_2x_2 + \ldots + r_nx_n \geq r_m$

$x_1 + x_2 + \ldots x_n = 1.0$

where:

r_i = expected annual return on investment i

r_m = the minimum desired annual return from the portfolio

Investment Portfolio Selection Example Problem Solution Using Excel (1 of 5)

Four stocks, desired annual return of at least 0.11.

Minimize

$$Z = S = x_A^2(.009) + x_B^2(.015) + x_C^2(.040) + X_D^2(.023) +$$
$$x_Ax_B\,(.4)(.009)^{1/2}(0.015)^{1/2} + x_Ax_C(.3)(.009)^{1/2}(.040)^{1/2} +$$
$$x_Ax_D(.6)(.009)^{1/2}(.023)^{1/2} + x_Bx_C(.2)(.015)^{1/2}(.040)^{1/2} +$$
$$x_Bx_D(.7)(.015)^{1/2}(.023)^{1/2} + x_Cx_D(.4)(.040)^{1/2}(.023)^{1/2} +$$
$$x_Bx_A(.4)(.015)^{1/2}(.009)^{1/2} + x_Cx_A(.3)(.040)^{1/2} + (.009)^{1/2} +$$
$$x_Dx_A(.6)(.023)^{1/2}(.009)^{1/2} + x_Cx_B(.2)(.040)^{1/2}(.015)^{1/2} +$$
$$x_Dx_B(.7)(.023)^{1/2}(.015)^{1/2} + x_Dx_C\,(.4)(.023)^{1/2}(.040)^{1/2}$$

subject to:

$$.08x_1 + .09x_2 + .16x_3 + .12x_4 \geq 0.11$$
$$x_1 + x_2 + x_3 + x_4 = 1.00$$
$$x_i \geq 0$$

Investment Portfolio Selection Example Problem Solution Using Excel (2 of 5)

Stock (x_i)	Annual Return (r_i)	Variance (s_i)
Altacam	.08	.009
Bestco	.09	.015
Com.com	.16	.040
Delphi	.12	.023

Stock combination (i,j)	Correlation (r_{ij})
A,B	.4
A,C	.3
A,D	.6
B,C	.2
B,D	.7
C,D	.4

Investment Portfolio Selection Example Problem Solution Using Excel (3 of 5)

Microsoft Excel - Exhibit10.13

B20

```
=(E6^2)*C6+(E7^2)*C7+(E8^2)*C8+(E9^2)*C9+2*(E6*E7*C14*SQRT(C6)*SQRT(C7)+E6*E8*D14*SQRT(C6)*
SQRT(C8)+E6*E9*E14*SQRT(C6)*SQRT(C9)+E7*E8*D15*SQRT(C7)*SQRT(C8)+E7*E9*E15*SQRT(C7)*
SQRT(C9)+E8*E9*E16*SQRT(C8)*SQRT(C9))
```

Doubling covariances will include all investment pairs.

Jessica Todd's Investment Portfolio

Stock	Average Return	Estimated Variance	Variables	Investment Proportion
Altacam	0.08	0.009	X1 =	
Bestco	0.09	0.015	X2 =	
Com.com	0.16	0.040	X3 =	
Delphi	0.12	0.023	X4 =	
Total desired return =	0.11		*Sum =*	1.000

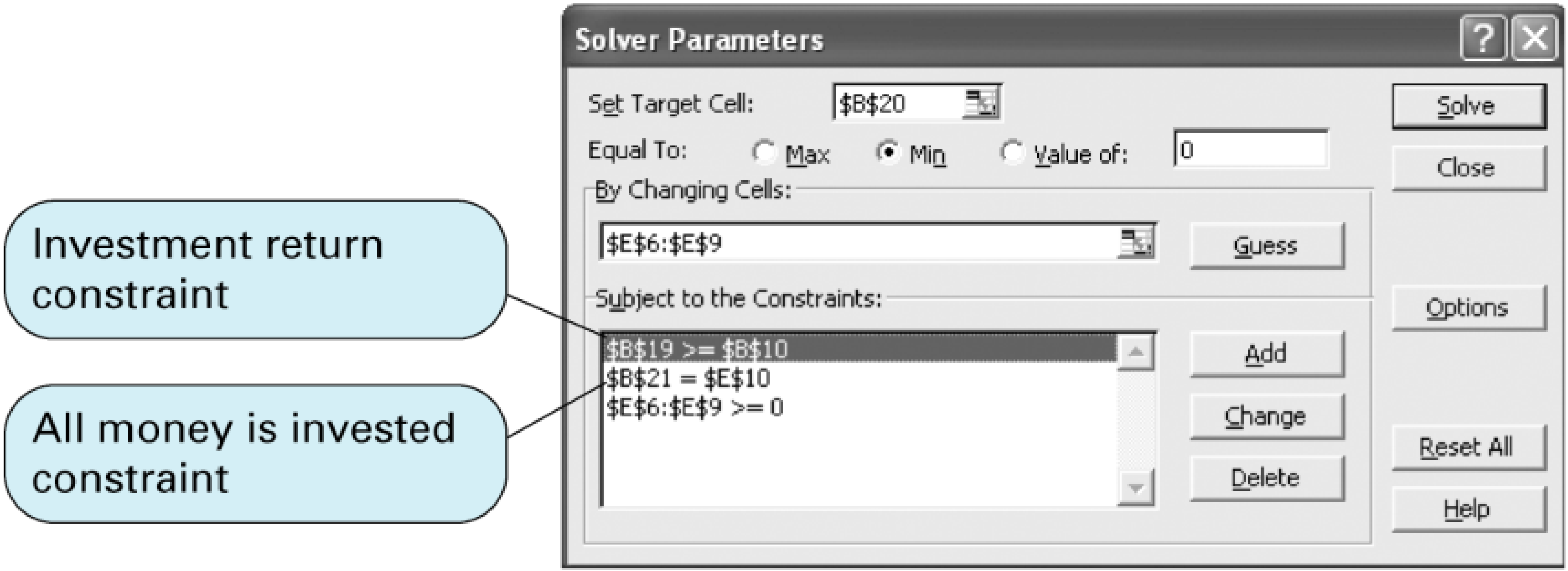
Correlations:				
Stock	Altacam	Bestco	Com.com	Delphi
Altacam	1	0.40	0.30	0.60
Bestco	0.40	1	0.20	0.70
Com.com	0.30	0.20	1	0.40
Delphi	0.60	0.70	0.40	1

Total portfolio return =	0
Total variance =	0
Total proportion invested =	0

=SUM(E6:E9)

=SUMPRODUCT(B6:B9,E6:E9)

Investment Portfolio Selection Example Problem Solution Using Excel (4 of 5)



Investment Portfolio Selection Example Problem Solution Using Excel (5 of 5)

Microsoft Excel - Ch10-NLPPortfolio.xls

B20

```
=(E6^2)*C6+(E7^2)*C7+(E8^2)*C8+(E9^2)*C9+2*(E6*E7*C14*SQRT(C6)*SQRT(C7)+E6*E8*D14*
SQRT(C6)*SQRT(C8)+E6*E9*E14*SQRT(C6)*SQRT(C9)+E7*E8*C15*SQRT(C7)*SQRT(C8)+E7*E9*
E15*SQRT(C7)*SQRT(C9)+E8*E9*E16*SQRT(C8)*SQRT(C9))
```

Jessica Todd's Investment Portfolio

Stock	*Average Return*	*Estimated Variance*	*Variables*	*Investment Proportion*
Altacam	0.08	0.009	X1 =	0.532
Bestco	0.09	0.015	X2 =	0.000
Com.com	0.16	0.040	X3 =	0.282
Delphi	0.12	0.023	X4 =	0.185
Total desired return =	0.11		*Sum =*	1.000

Covariances:

Stock	Altacam	Bestco	Com.com	Delphi
Altacam	1	0.40	0.30	0.60
Bestco	0.40	1	0.20	0.70
Com.com	0.30	0.20	1	0.40
Delphi	0.60	0.70	0.40	1

Total portfolio return =	0.11
Total variance =	0.0112
Total proportion invested =	1.00

Quick Review Question

Follow Up Assignment

Summary of Main Teaching Points

Question and Answer Session

Q & A

Next Lesson

Network Models