Operational Research and Optimisation

AQ052-3-M-ORO and VD1

Waiting Line Models

Topic & Structure of the lesson

- The Structure of a Waiting Line System
- Queuing Systems
- Queuing System Input Characteristics
- Queuing System Operating Characteristics
- Single-Channel Waiting Line Model with Poisson Arrivals and Exponential Service Times
- Multiple-Channel Waiting Line Model with Poisson Arrivals and Exponential Service Times

Learning Outcomes

- **A the end of this topic, You should be able to model and evaluate a waiting line problem.**

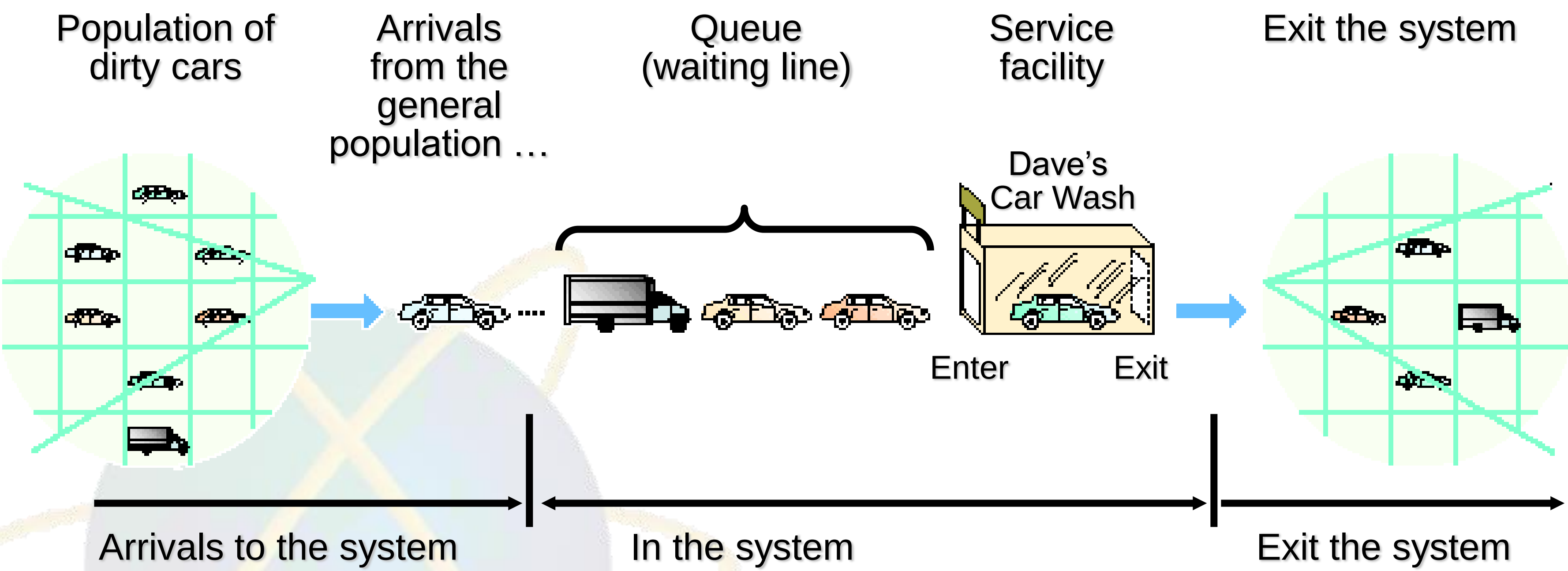
Key Terms you must be able to use

If you have mastered this topic, you should be able to use the following terms correctly in your assignments and exams:

Common Queuing Situations

Situation	*Arrivals in Queue*	*Service Process*
Supermarket	***Grocery shoppers***	***Checkout clerks at cash register***
Doctor's office	***Patients***	***Treatment by doctors and nurses***
Computer system	***Programs to be run***	***Computer processes jobs***
Telephone company	***Callers***	***Switching equipment to forward calls***
Bank	***Customer***	***Transactions handled by teller***
Machine maintenance	***Broken machines***	***Repair people fix machines***

Parts of a Waiting Line Model



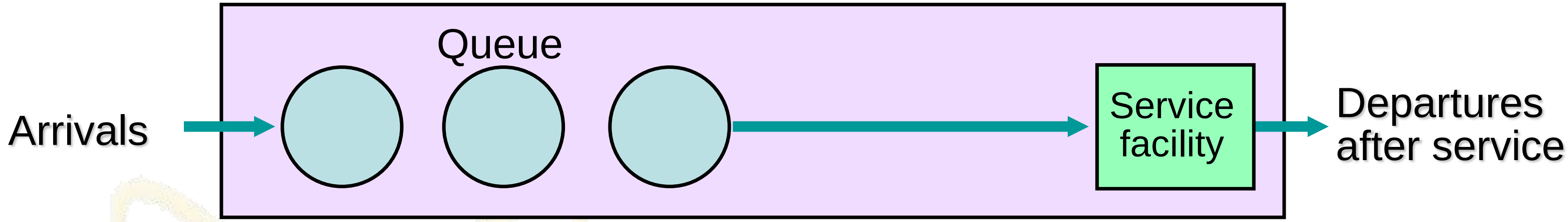
Arrival Characteristics
- ☑ Behavior of arrivals
- ☑ Statistical distribution of arrivals

Waiting Line Characteristics
- ☑ Limited (finite) vs. unlimited (infinite)
- ☑ Queue discipline

Service Characteristics
- ☑ Service design
- ☑ Statistical distribution of service

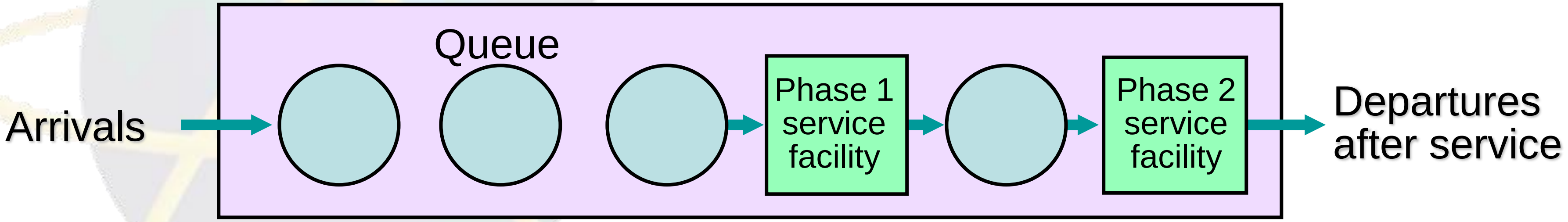
Queuing System Designs

A family dentist's office



Single-channel, single-phase system

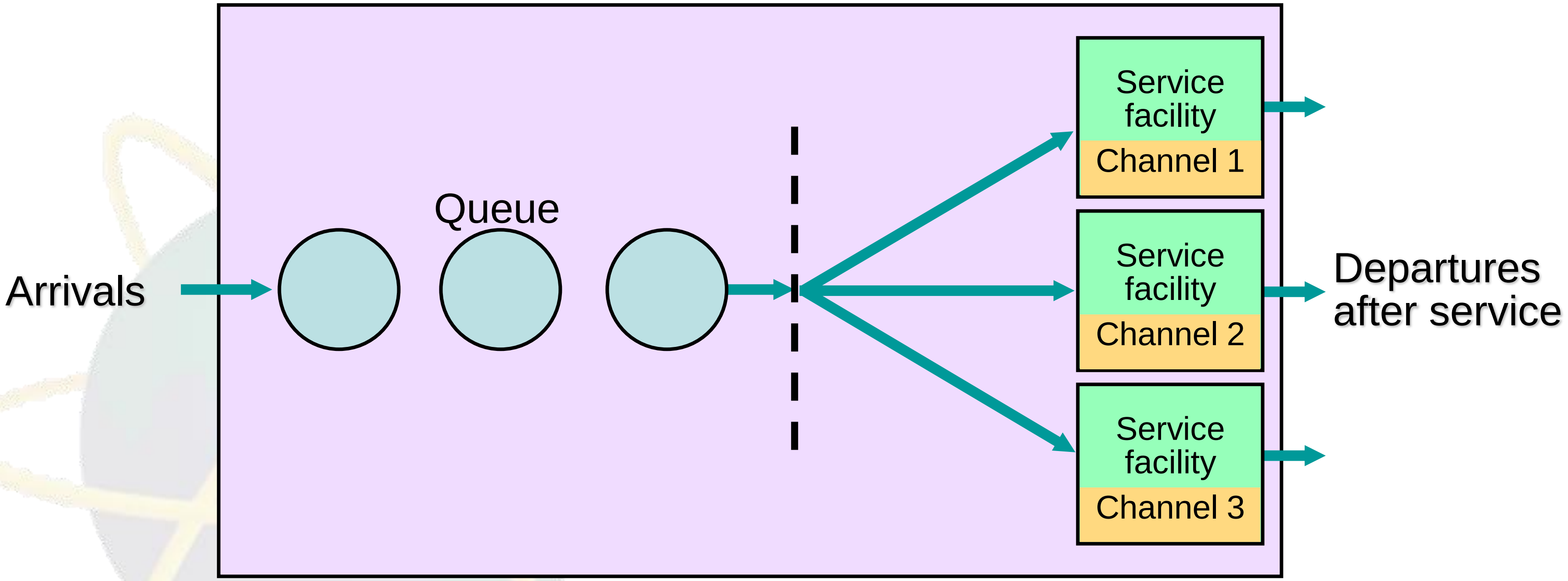
A McDonald's dual window drive-through



Single-channel, multiphase system

Queuing System Designs

Most bank and post office service windows

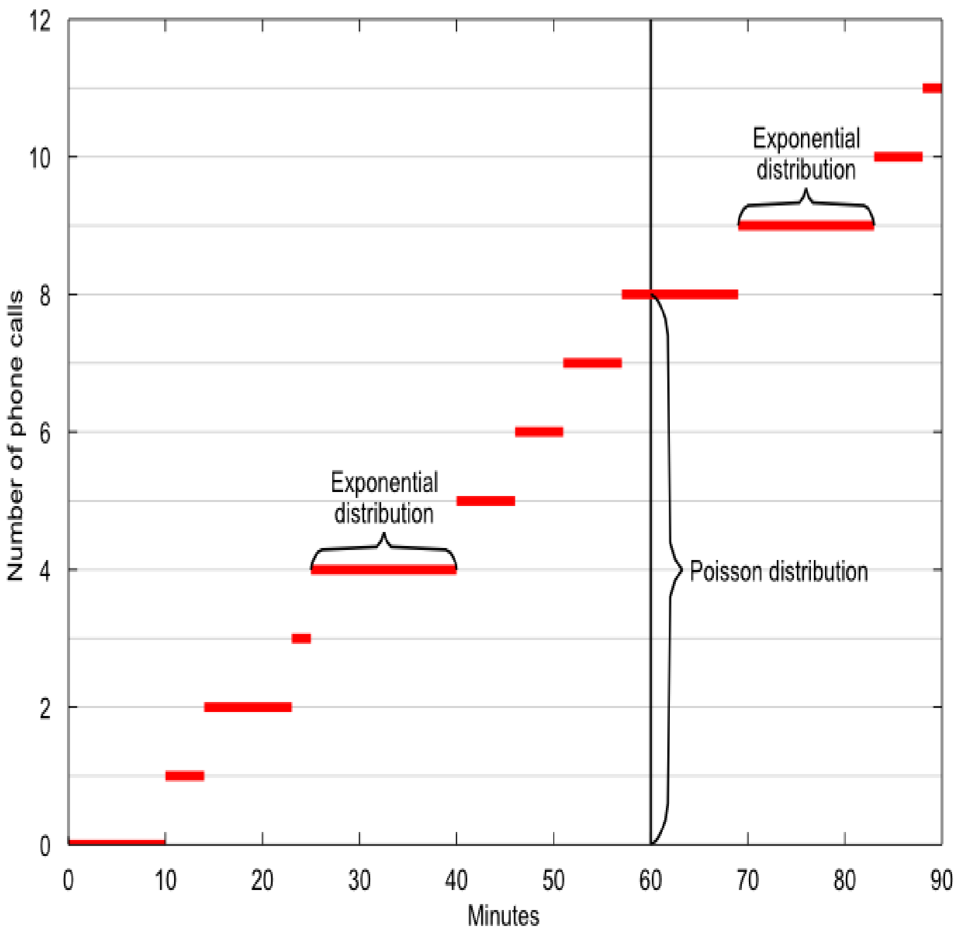


Multi-channel, single-phase system

Structure of a Waiting Line System

- Distribution of Arrivals
  - Generally, the arrival of customers into the system is a random event.
  - Frequently the arrival pattern is modeled as a Poisson process.
  - Eg: What is the probability of having 30 people to visit the restaurant in 1-hour if the mean of people to visit is 2 per 5 minutes?
- Distribution of Service Times
  - Service time is also usually a random variable.
  - A distribution commonly used to describe service time is the exponential distribution.
  - Eg: What is the probability that a customer has to wait more than 3 minutes to get the service?

What is Poisson? Exponential?



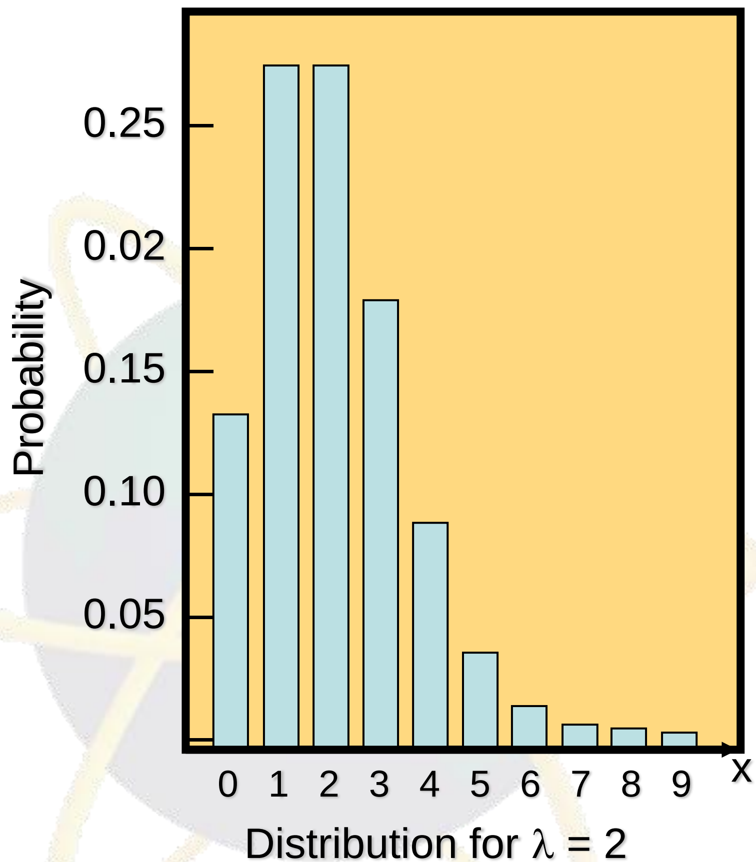
A classical example of a random variable having a Poisson distribution is the number of phone calls received by a call center.
If the time elapsed between two successive phone calls has an exponential distribution and it is independent of the time of arrival of the previous calls, then the total number of calls received in one hour has a Poisson distribution.

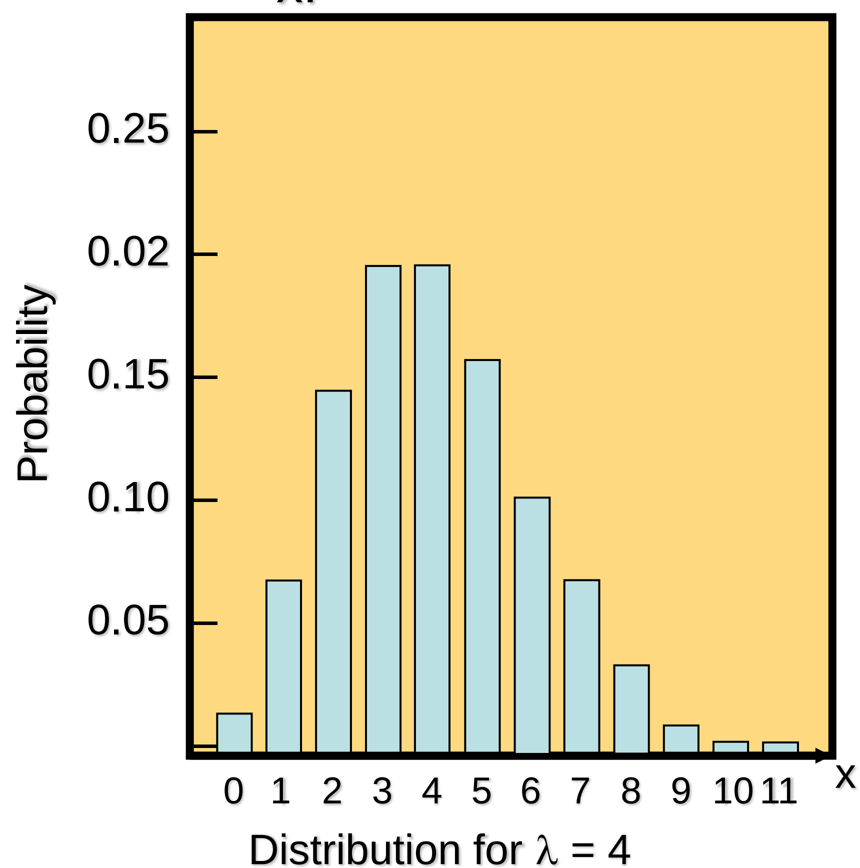
Example

- If job arrives every 15 seconds on average, $\lambda = 4$ per minute, what is the probability of waiting less than or equal to 30 seconds, ie. 0.5 min?
  – (Ans: 0.86)
- Accident occur with a Poisson distribution at an average of 4 per week. Calculate the probability of more than 5 accidents in any one week.
  – (Ans: 0.215)

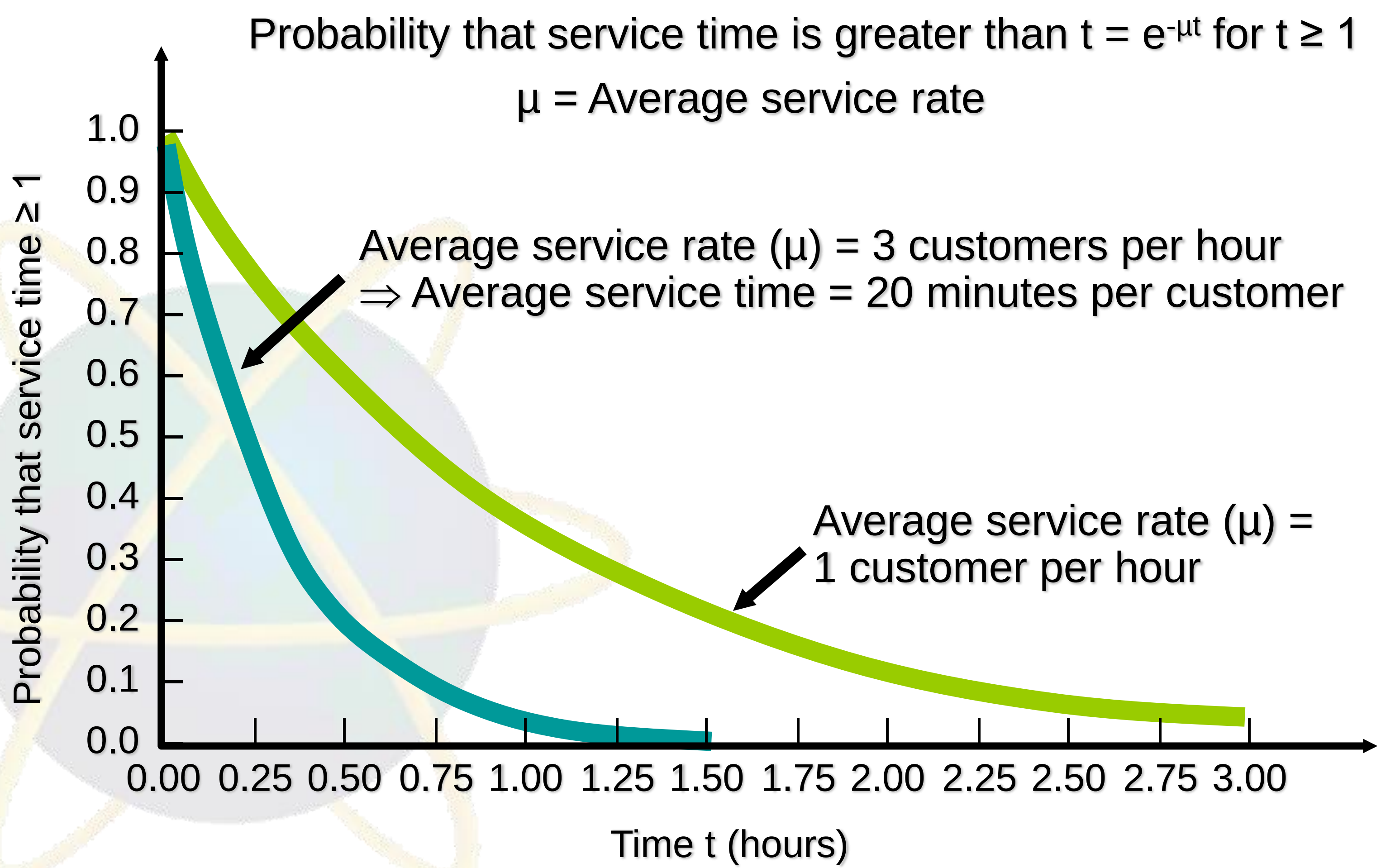
Poisson Distribution

$$\text{Probability} = P(x) = \frac{e^{-\lambda}\lambda^x}{x!}$$





Exponential Distribution



Queuing Systems

- A three part code of the form *A*/*B*/*k* is used to describe various queuing systems.
- *A* identifies the arrival distribution, *B* the service distribution and *k* the number of channels for the system.
- Symbols used for the arrival and service processes are:

  ***M* - Markov distributions (Poisson/exponential)**

  D - Deterministic (constant) and

  G - General distribution (with a known mean and variance).
- For example, *M*/*M*/*k* refers to a system in which arrivals occur according to a Poisson distribution, service times follow an exponential distribution and there are *k* servers working at identical service rates.

Queuing System Input Characteristics

λ = the average arrival <u>rate</u>

$1/\lambda$ = the average <u>time</u> between arrivals

μ = the average service <u>rate</u> for each server

$1/\mu$ = the average service <u>time</u>

σ = the standard deviation of the service <u>time</u>

Queuing System Operating Characteristics

P_0 = probability the service facility is idle

P_n = probability of n units in the system

P_w = probability an arriving unit must wait for service

L_q = average number of units in the queue awaiting service

L = average number of units in the system

W_q = average time a unit spends in the queue awaiting service

W = average time a unit spends in the system

Queuing Models

Model	Name	Example
A	Single-channel system (M/M/1)	Information counter at department store

Number of Channels	Number of Phases	Arrival Rate Pattern	Service Time Pattern	Population Size	Queue Discipline
Single	Single	Poisson	Exponential	Unlimited	FIFO

Queuing Models

Model	Name	Example
B	Multichannel (M/M/k)	Airline ticket counter

Number of Channels	Number of Phases	Arrival Rate Pattern	Service Time Pattern	Population Size	Queue Discipline
Multi-channel	Single	Poisson	Exponential	Unlimited	FIFO

Queuing Models

Model	Name	Example
C	Limited population (finite population)	Shop with only a dozen machines that might break

Number of Channels	Number of Phases	Arrival Rate Pattern	Service Time Pattern	Population Size	Queue Discipline
Single	Single	Poisson	Exponential	Limited	FIFO

M/M/1 Queuing System

- Single channel
- Poisson arrival-rate distribution
- Exponential service-time distribution
- Unlimited maximum queue length
- Infinite calling population
- Examples:
  - Single-window theatre ticket sales booth
  - Single-scanner airport security station

Example: SJJT, Inc. (A)

- *M*/*M*/1 Queuing System

  Joe Ferris is a stock trader on the floor of the New York Stock Exchange for the firm of Smith, Jones, Johnson, and Thomas, Inc. Stock transactions arrive at a mean rate of 20 per hour. Each order received by Joe requires an average of two minutes to process.

λ = 20/hr. ; μ = 30/hr.

Example: SJJT, Inc. (A)

- Average Time in the System

Question

What is the average time an order must wait from the time Joe receives the order until it is finished being processed (i.e. its turnaround time)?

Answer

This is an *M*/*M*/1 queue with λ = 20 per hour and μ = 30 per hour. The average time an order waits in the system is:

$$W = 1/(\mu - \lambda)$$
$$= 1/(30 - 20)$$
$$= 1/10 \text{ hour or } 6 \text{ minutes}$$

Example: SJJT, Inc. (A)

- Average Length of Queue

Question

What is the average number of orders Joe has waiting to be processed?

Answer

Average number of orders waiting in the queue is:

$$L_q = \lambda^2/[\mu(\mu - \lambda)]$$
$$= (20)^2/[(30)(30-20)]$$
$$= 400/300$$
$$= \boxed{4/3}$$

M/M/k Queuing System

- Multiple channels (with one central waiting line)
- Poisson arrival-rate distribution
- Exponential service-time distribution
- Unlimited maximum queue length
- Infinite calling population
- Examples:
  - Four-teller transaction counter in bank
  - Two-clerk returns counter in retail store

Example: SJJT, Inc. (B)

- *M/M/*2 Queuing System

The company has begun a major advertising campaign which it believes will increase its business 50%. To handle the increased volume, the company has hired an additional floor trader, Fred Hanson, who works at the same speed as Joe Ferris.

Note that the new arrival rate of orders, λ , is 50% higher than the previous. Thus, λ = 30 per hour.

Example: SJJT, Inc. (B)

- Sufficient Service Rate

Question

Why will Joe Ferris alone not be able to handle the increase in orders?

Answer

Since Joe Ferris processes orders at a mean rate of $\mu = 30$ per hour, then $\lambda = \mu = 30$ and the utilization factor is 1.

This implies the queue of orders will grow infinitely large. Hence, Joe alone cannot handle this increase in demand.

Example: SJJT, Inc. (B)

- Probability of *n* Units in System

  For the new system with both Joe and Fred,

  Question:

- What is the probability that there is no customer in the system?
- What is the average number of customers waiting in the line?
- What is the average number of customers in the system?
- What is the average time customer spent waiting in line?
- What is the average time customer spent in the system?

Example: SJJT, Inc. (C)

- Economic Analysis of Queuing Systems

  The advertising campaign of Smith, Jones, Johnson and Thomas, Inc. was so successful that business actually doubled. The mean rate of stock orders arriving at the exchange is now 40 per hour and the company must decide how many floor traders to employ. Each floor trader hired can process an order in an average time of 2 minutes. Based on a number of factors the brokerage firm has determined the average waiting cost per minute for an order to be $.50. Floor traders hired will earn $20 per hour in wages and benefits. Using this information compare the total hourly cost of hiring 2 traders with that of hiring 3 traders.

Example: SJJT, Inc. (C)

- Economic Analysis of Waiting Lines

Total Hourly Cost

= (Total salary cost per hour)

+ (Total hourly cost for orders in the system)

= ($20 per trader per hour) x (Number of traders)

+ ($30 waiting cost per hour) x

(Average number of orders in the system)

= 20k + 30L.

Thus, L must be determined for k = 2 traders and for k = 3 traders with λ = 40/hr. and μ = 30/hr. (since the average service time is 2 minutes (1/30 hr.).

Example: SJJT, Inc. (C)

- System Cost Comparison

	Wage Cost/Hr	Waiting Cost/Hr	Total Cost/Hr
2 Traders	$40.00	$82.00	$112.00
3 Traders	60.00	44.35	104.35

Thus, the cost of having 3 traders is less than that of 2 traders.

M/M/1 Queuing System with Finite Population

- Single channel
- Poisson arrival-rate distribution
- Exponential service-time distribution
- Finite calling population
- Examples:
  - Machine repair problem: if a machine breaks down before the pair work has been completed on the first machine, the second machine begins to form a “waiting line” for repair service.

Example

Five administrative assistants use an office copier. The average time between arrivals for each assistant is 40 minutes, which is equivalent to an arrival rate of 0.025 arrivals per minute. The mean time each assistant spends at the copier is 5 minutes, which is equivalent to a service rate of 0.20 minutes. Use M/M/1 model with finite population to determine:

(i) Probability that the copier is idle

(ii) Average time an assistant spends waiting for the copier.

(iii) Average time an assistant spends at the copier.

(iv) Should management consider purchasing a second copier, assuming they are working for 8-hour day?

Quick Review Question

Follow Up Assignment

Summary of Main Teaching Points

Question and Answer Session

Q & A

Next Session

Simulation